

SUPERFLUID TO NORMAL PHASE TRANSITION AND EXTREME REGULARITY OF SUPERDEFORMED BANDS

I. M. Pavlichenkov

January 23, 2002

Abstract

We derive the exact semiclassical expression for the second inertial parameter \mathcal{B} for the superfluid and normal phases. Interpolation between these limiting values shows that the function $\mathcal{B}(I)$ changes sign at the spin I_c , which is critical for a rotational spectrum. The quantity \mathcal{B} turns out to be a sensitive measure of the change in static pairing correlations. The superfluid-to-normal transition reveals itself in the specific variation of the ratio \mathcal{B}/\mathcal{A} versus spin I with the plateau characteristic of the normal phase. We find this dependence to be universal for normal deformed and superdeformed bands. The long plateau with a small value $\mathcal{B}/\mathcal{A} \sim A^{-8/3}$ explains the extreme regularity of superdeformed bands.

1 Introduction

Recently phase transitions in mesoscopic systems have been a subject of intense discussions in nuclear and solid-state physics. Isai Isidorovitch Gurevich appears to be the first to bring up (in 1939) the concept of the temperature phase transition to nuclear studies [1]. His prediction was based on the observation that the level density of the resonant states formed by a thermal neutron capture is an unsteady function of the atomic mass number with a maximum in the rare-earth nuclei.

The problem of the rotation-induced transition from the superfluid to the normal phase in nuclei has been a foremost theme in high-spin spectroscopy since Mottelson and Valatin [2] predicted a pairing collapse in rapidly rotating nuclei. This effect can be understood by an analogy with a superconductor in a magnetic field. In a deformed nucleus the Cooper pair is formed by two nucleons with opposite single-particle angular momentum projections $\pm m$. Being time noninvariant (as a magnetic field) the Coriolis force in a rotating nucleus acts on both nucleons in opposite directions and tries to decrease the spatial overlap of these time reversal orbits. The Coriolis force increases proportionally with the spin of a band. Therefore, at some critical spin one may expect that all pairs are broken and pairing correlations disappear completely. The phenomenon can be observed by the crossing of the ground state superfluid band with the band based on the normal state. Thus, the rigid body moment of inertia corresponding to the second band appears to be the obvious signature of the pairing phase transition.

However, this regime of the transition to the normal phase is not realized in nuclei because they are finite systems with a shell structure and a small number of nucleons involved in pairing correlations. The Coriolis force in a rotating nucleus is proportional to the value of the single-particle angular momentum j of a nucleon. Thus, the Coriolis antipairing effect is strongest for nucleons occupying the states of subshells with the largest j . In the vicinity of the Fermi surface, these so-called intruder orbitals arise from the $j = \mathcal{N} + 1/2$ subshell, where \mathcal{N} is the principal quantum number of the shell above. Therefore they are distinguished from other states of the unfilled shell by the parity. At normal deformations (ND), the intruder states retain their j quantum number, while at superdeformations (SD) the j -subshell notation becomes less appropriate due to mixing. Initially the Coriolis force breaks only the one Cooper pair that belongs to the intruder orbitals, whereas the rest of the pairs stay correlated. The band built on such two-quasiparticle excitation (the rotationally-aligned band) is characterized, due to the blocking effect, by appreciably smaller pairing correlations than in the ground state band. Having the largest moment of inertia, the former crosses the ground state band and becomes yrast. The relevant phenomenon is backbending. Subsequent breaking of correlated pairs and their alignment makes the internal structure of the yrast band non-homogeneous and the transition to the normal phase configuration dependent.

The standard definition of the phase transition is based on the mean-field approximation in which different phases are distinguished by the order parameter, i. e., the static pairing gap Δ . However the mean field approach to the nuclear pairing correlations faces a fundamental problem of the quantum fluctuations, which become quite strong for finite systems. The fluctuating part $\delta\Delta$ (dynamical pairing correlations) of the order parameter is comparable with Δ in the transition region. The fluctuations smear out a sharp phase transition and make it a difficult issue to find the experimental signature of the superfluid to normal phase transition in rotational bands. For example, the dependence of the kinematic

$\mathfrak{S}^{(1)}$ or dynamical $\mathfrak{S}^{(2)}$ moments of inertia on the spin I is not a definite indicator of the phase transition. Experimental evidence of the pairing phase transitions has been discussed usually in terms of the relative excitation spectrum. As shown in Ref. [3] the disappearance of static pairing leads to a change of the excitation spectrum, from the quasiparticle to the particle-hole spectrum. Unfortunately, the application of this criterion to normal deformed (ND) bands [3, 4, 5] shows that this method is not free from ambiguities.

Meanwhile, it is well known that the change in the system internal structure manifests itself in the modification of its collective excitation. The examples for finite quantum systems are numerous. A classical one is the transition from deformed to spherical nuclei. In this case, the rotational-vibrational spectrum transforms into a pure vibrational one. The study of the bifurcations in rotational spectra [6] shows an intimate connection between internal and rotational motion. For example, the angular momentum alignment in a band (the change of a coupling scheme) is observed as an increase of the energy signature splitting [7].

The transition we study is more delicate. Consider the simplest rotational sequence with the parity and the signature $(\pi\alpha) = (+0)$. The relevant energy spectrum can be parametrized as follows:

$$E(I) = \mathcal{A}I(I+1) + \mathcal{B}I^2(I+1)^2, \quad (1.1)$$

where $\mathcal{A} = \hbar^2/2\mathfrak{S}^{(1)}$ and \mathcal{B} are the first two inertial parameters. The spectrum (1.1) undergoes a noticeable modification if, for example, the second inertial parameter changes sign at some spin I_c . For $I < I_c$ the spectrum is compressed relative to the rigid rotor spectrum because \mathcal{B} is negative for the low- I states. However, for $I > I_c$ the spectrum becomes extended. The effect can be visualized by using the I dependence of the ratio \mathcal{B}/\mathcal{A} . The main objective of the present paper is to study this dependence.

The parameters \mathcal{A} and \mathcal{B} are determined by the γ -ray transition energies $E_\gamma(I) = E(I+2) - E(I)$ as follows:

$$\begin{aligned} \mathcal{A}(I) &= \frac{1}{4(2I+5)} \left[\frac{I^2+7I+13}{2I+3} E_\gamma(I) - \frac{I^2+3I+3}{2I+7} E_\gamma(I+2) \right], \\ \mathcal{B}(I) &= \frac{1}{8(2I+5)} \left[\frac{E_\gamma(I+2)}{2I+7} - \frac{E_\gamma(I)}{2I+3} \right]. \end{aligned} \quad (1.2)$$

The coefficient \mathcal{B} characterizes the nonadiabatic properties of a band and is very sensitive to its internal structure. It also realizes the relationship between kinematic and dynamic moments of inertia. Using the well-known expressions for these quantities (see, for example, Ref. [8]) and the last formula (1.2) we get

$$\mathcal{B} = \frac{\hbar^2}{2(2I+3)(2I+7)} \left[\frac{1}{\mathfrak{S}^{(2)}} - \frac{2I}{(2I+5)\mathfrak{S}^{(1)}} \right]. \quad (1.3)$$

Thus, the parameter $\mathcal{B}(I)$ is proportional to the difference $\mathfrak{S}^{(1)} - \mathfrak{S}^{(2)}$ in the high- I limit. The ratio \mathcal{B}/\mathcal{A} also determines the convergence radius of the rotational energy expansion in terms of $I(I+1)$ [9]. Faster convergence is obtained with the Harris formula

$$E(\omega) = E_0 + \frac{1}{2}\alpha\omega^2 + \frac{3}{4}\beta\omega^4 + \dots, \quad (1.4)$$

which is based on the fourth-order cranking expansion

$$\alpha = \frac{1}{\omega} \text{tr}(j_x \rho^{(1)}), \quad \beta = \frac{1}{\omega^3} \text{tr}(j_x \rho^{(3)}), \quad (1.5)$$

where $\rho^{(n)}$ is the n th correction to the nucleus density matrix; j_x is the projection of the single-particle angular momentum operator onto the rotational axis x , which is perpendicular to the symmetry axis z ; and ω is the rotational frequency. The latter depends on the system angular momentum and is determined by

$$\hbar\sqrt{I(I+1)} = \alpha\omega + \beta\omega^3 + \dots \quad (1.6)$$

It follows from Eqs. (1.1), (1.4), and (1.6) that

$$\alpha = \frac{\hbar^2}{2\mathcal{A}}, \quad \beta = -\frac{\hbar^4\mathcal{B}}{4\mathcal{A}^4}. \quad (1.7)$$

The problem of the microscopic calculation of the parameter \mathcal{B} for ND nuclei has attracted considerable attention (see the review article [10] and references therein). It has been shown that this quantity receives the contributions from four types of nonadiabatic effects:

- (i) perturbation of quasiparticle motion by rotation (quasiparticle alignment),
- (ii) attenuation of pairing correlations by the Coriolis force (Coriolis antipairing effect),
- (iii) a change in the deformation of nuclear self-consistent field (centrifugal stretching effect),
- (iv) vibration-rotation interaction.

The first attempt to estimate \mathcal{B} was made by the author together with Grin' [11]. A Green's function formulation of the Hartree-Fock-Bogolubov (HFB) method was used to find the expansion (1.4) for the axially deformed oscillator potential as the self-consistent field. It was shown that the first and the second effects yield $\mathcal{B}/\mathcal{A} \sim A^{-4/3}$ while the centrifugal stretching contribution is $A^{2/3}$ times smaller for well deformed nuclei.¹ In the subsequent work [12], the author found that the vibration-rotation contribution to the parameter \mathcal{B} accounts for the same $A^{2/3}$ fraction of the main effects. These results were confirmed by the calculations of Marshalek [13] with the more realistic Nilsson potential.

Thus, the first two effects are dominant for well deformed nuclei. The quasiparticle alignment depends strongly on pairing correlations because the pairing force tries to bind pairs of particles in time reversal states, reducing the ability of nucleons to carry an angular momentum. Therefore, the parameter \mathcal{B} is very sensitive to the variation of the pairing correlations along a band.

One of the amazing features of SD bands is the extreme regularity of their rotational spectra. To demonstrate this feature, the rotational spectra of different axial systems are compared in Fig. 1 with their rigid rotor counterparts. The comparison shows that the SD band $^{194}\text{Pb}(1)$ is more regular than the ND band of ^{238}U and even the band of the simplest H_2 molecule [14]. Having the ratio $\mathcal{B}/\mathcal{A} \sim 10^{-5}$, the band $^{194}\text{Pb}(1)$ is not a champion among SD bands. For $^{152}\text{Dy}(1)$, the ratio is of the order 10^{-6} and this is 1000 times smaller than the above estimation $\mathcal{B}/\mathcal{A} \sim A^{-4/3}$. Thus, an SD nucleus is the best quantum rotor known in nature. Although numerous theoretical calculations (see, e.g., [15, 16, 17, 18]) successfully reproduce the measured intraband γ -ray energies, the underlying microscopic mechanism of this phenomenon has yet to be well understood.

In this paper we will reveal an interconnection between the extreme regularity and the transition from the superfluid to the normal phase. The key to our theoretical approach lies in the calculation of the second inertial parameter. Compared to ND bands, there are two features of the pairing correlations in SD ones which prevent us from using the results

¹We use the estimation $\mathcal{A} \sim \varepsilon_F A^{-5/3}$, where ε_F is the Fermi energy and A is the mass number.

of previous theoretical calculations of the parameter \mathcal{B} for superdeformation. First, due to the large shell gap stabilizing the SD minimum, the static pairing field Δ is small and can be commensurate with its fluctuation $\delta\Delta$. Qualitative conclusion concerning the role of the static and dynamic pairing in SD bands is presented in Ref. [19]. Second, since intruder single-particle states, which are unavailable at normal deformations, appear near the Fermi surface in the case of superdeformations, it is necessary to go beyond the commonly used monopole pairing force [20]. The gauge invariant pairing interaction expands the correlation space and stabilizes the pairing field. The coordinate dependent (nonuniform) pairing is also crucial for conservation of a nucleon current in a rotating nucleus [21].

To avoid calculation of the parameter \mathcal{B} in the transition region, where pairing fluctuations play an important role, an interpolation between the values \mathcal{B}_s and \mathcal{B}_n is used. The former is associated with the superfluid phase (where $\Delta \gg \delta\Delta$) and the latter is related to the normal one ($\Delta = 0$). Thus, pairing fluctuations are unessential for these regions and we can use the mean-field approach. In the calculation of \mathcal{B}_s , the nonuniform pairing induced by rotation is taken into account by using the method of Ref. [11]. It should be noted that the quantity \mathcal{B}_s found in the cited work is inapplicable for superdeformation.

The paper is organized as follows: In Section 2, the basic equations of the cranked HFB theory are presented in the framework of the Green's function method. The spinor form of the Gor'kov equations is used to simplify calculations in the higher orders of the perturbation theory. In Section 3, the exact expression for the second inertial parameter in the superfluid phase is derived with this technique by applying a semiclassical approximation. The result is valid for an arbitrary nuclear mean field. The exact analytical expression for \mathcal{B}_s is obtained in Section 4 in an axially deformed oscillator potential. In this section we also consider some limiting cases for this quantity. Of special interest is the limit of noninteracting nucleons. It is shown that the relevant parameter \mathcal{B}_n is positive and smaller than \mathcal{B}_s . The comparison with available experimental data for SD and ND bands is presented in Section 5. Section 6 concludes and summarizes the paper. The preliminary results of the present work have been published in Refs. [22, 23].

2 Green's function formalism in the cranking Hartree-Fock-Bogolubov method

2.1 Cranked Gor'kov equations

Our consideration is based on the shell-model Hamiltonian consisting of the cranked one body term

$$h_\omega(\mathbf{r}) = -\frac{\mathbf{p}^2}{2M} + U(\mathbf{r}) - \omega \cdot \ell, \quad \omega \in \{\omega, 0, 0\} \quad (2.1)$$

(where \mathbf{p} and M are the impulse and the mass of a nucleon respectively), and the residual short-range interaction, which is specified by the two body delta-interaction

$$v(\mathbf{r}, \mathbf{r}') = -g\delta(\mathbf{r} - \mathbf{r}'), \quad g > 0. \quad (2.2)$$

For simplicity, we neglect the spin in the cranking term and consider only the orbital part ℓ of the angular momentum \mathbf{j} . We will also neglect a weak dependence of the self-consistent deformed potential U on rotation (centrifugal stretching effect).

In the coordinate representation, the Gor'kov equations [24] have the form

$$\begin{aligned} [\varepsilon - h_\omega(\mathbf{r}) + \varepsilon_F]G(\mathbf{r}, \mathbf{r}', \varepsilon) + \tilde{\Delta}(\mathbf{r})F^+(\mathbf{r}, \mathbf{r}', \varepsilon) &= \delta(\mathbf{r} - \mathbf{r}'), \\ [\varepsilon + h_\omega^+(\mathbf{r}) - \varepsilon_F]F^+(\mathbf{r}, \mathbf{r}', \varepsilon) + \tilde{\Delta}^+(\mathbf{r})G(\mathbf{r}, \mathbf{r}', \varepsilon) &= 0, \\ \tilde{\Delta}^*(\mathbf{r}) &= g \oint_C \frac{d\varepsilon}{2\pi i} F^+(\mathbf{r}, \mathbf{r}, \varepsilon). \end{aligned} \quad (2.3)$$

The functions $G(\mathbf{r}, \mathbf{r}', \varepsilon)$ and $F^+(\mathbf{r}, \mathbf{r}', \varepsilon)$ are the Fourier transforms of the Green's functions

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}', t - t') &= -i(\Phi_N | T\{\psi(\mathbf{r}, t)\psi^+(\mathbf{r}', t)\} | \Phi_N), \\ F^+(\mathbf{r}, \mathbf{r}', t - t') &= -i(\Phi_{N+2} | T\{\psi^+(\mathbf{r}, t)\psi^+(\mathbf{r}', t)\} | \Phi_N) e^{-2i\varepsilon_F t}, \end{aligned} \quad (2.4)$$

where Φ_N and Φ_{N+2} are the eigenfunctions of the ground state of a system of N and $N + 2$ interacting particles, ψ^+ and ψ are creation and annihilation operators in the Heisenberg representation, T is the time ordering operator, and ε_F is the Fermi energy (the chemical potential of the system). The contour C consists of the real axis and an infinite semicircle in the upper half-plane.

2.2 Properties

In obtaining Eqs. (2.3) the particle number nonconserving approximation has been used. In a spirit of the mean-field approach we neglect a difference between the functions Φ_N and Φ_{N+2} . Thus, the Gor'kov equations describe a system with the broken gauge symmetry associated with the particle number. However, the average particle number is fixed. This is achieved by adding the term $-\varepsilon_F \hat{N}$ to the Hamiltonian. The Lagrange multiplier ε_F is determined by the equation

$$N = \int d\mathbf{r} \oint_C \frac{d\varepsilon}{2\pi i} G(\mathbf{r}, \mathbf{r}, \varepsilon). \quad (2.5)$$

Equations (2.3) are also noninvariant with respect to the more general gauge transformation (the local Gallileian transformation [25])

$$\psi(\mathbf{r}, t) \rightarrow \psi(\mathbf{r}, t) e^{i\phi(\mathbf{r})}, \quad (2.6)$$

where $\phi(\mathbf{r})$ is an arbitrary function of the space coordinates. The quickest way to show this is to introduce the vector potential $\mathbf{A} = [\omega\mathbf{r}]/2$ that allows us to rewrite the Coriolis term $V = -\omega \cdot \ell$ in the form $-2\mathbf{p} \cdot \mathbf{A}$. It is seen that the Hamiltonian (2.1) lacks the term $2M\mathbf{A}^2$ which is absolutely necessary for the gauge invariance of Eqs. (2.3). However, since the two-body interaction (2.2) is invariant under the Gallileian transformation the conservation of nucleon current is ensured.

The current density is expressed in terms of the Green's function G as follows [26]:

$$\mathbf{j}(\mathbf{r}) = \lim_{\mathbf{r}' \rightarrow \mathbf{r}} \oint_C \frac{d\varepsilon}{2\pi i} \left\{ \frac{i\hbar}{2M} (\nabla_{\mathbf{r}} - \nabla_{\mathbf{r}'}) - [\omega\mathbf{r}] \right\} G(\mathbf{r}, \mathbf{r}', \varepsilon). \quad (2.7)$$

With this definition, we find

$$\text{div } \mathbf{j}(\mathbf{r}) = \lim_{\mathbf{r}' \rightarrow \mathbf{r}} \oint_C \frac{d\varepsilon}{2\pi i} [h_\omega(\mathbf{r}) - h_\omega^+(\mathbf{r}')] G(\mathbf{r}, \mathbf{r}', \varepsilon). \quad (2.8)$$

Using the first of Eq. (2.3) and their complex conjugate equation

$$[\varepsilon - h_\omega^+(\mathbf{r}') + \varepsilon_F]G(\mathbf{r}, \mathbf{r}', \varepsilon) + \tilde{\Delta}^*(\mathbf{r}')F(\mathbf{r}, \mathbf{r}', \varepsilon) = \delta(\mathbf{r} - \mathbf{r}'), \quad (2.9)$$

we finally obtain

$$\text{div } \mathbf{j}(\mathbf{r}) = \oint_C \frac{d\varepsilon}{2\pi\hbar} [\tilde{\Delta}(\mathbf{r})F^+(\mathbf{r}, \mathbf{r}, \varepsilon) - \tilde{\Delta}^*(\mathbf{r})F(\mathbf{r}, \mathbf{r}, \varepsilon)]. \quad (2.10)$$

The right-hand side of this equality vanishes due to the third Gor'kov equation. Because the latter is derived assuming delta-interaction we should conclude that the form of a two-body interaction is essential for obtaining the current conservation. In particular, the commonly used monopole pairing interaction is not invariant under the transformation (2.6). Therefore it does not conserve the current density in a rotating nucleus. The case of an arbitrary pairing interaction is considered in Ref. [25].

2.3 Matrix form of the Gor'kov equations

The two-dimensional form of the Gor'kov equations is very convenient to use in our calculations. Let us introduce the second pair of the Green's functions $G^+(\mathbf{r}, \mathbf{r}', \varepsilon)$ and $F(\mathbf{r}, \mathbf{r}', \varepsilon)$. It is easily proved [11] that the four equations for these functions can be written in the matrix form

$$\begin{pmatrix} \tilde{\Delta}^* & \varepsilon + h_\omega^+ - \varepsilon_F \\ \varepsilon - h_\omega + \varepsilon_F & \tilde{\Delta} \end{pmatrix} \begin{pmatrix} F & G \\ G^+ & F^+ \end{pmatrix} = \hat{1}\delta(\mathbf{r} - \mathbf{r}'). \quad (2.11)$$

The operator h_ω involves the real h and the imaginary V parts. The former is the Hamiltonian of the deformed mean field, whereas the latter denotes the cranking term, $V = -\omega\ell_x$. Separating the quantity $\tilde{\Delta}$ into real and imaginary parts,

$$\tilde{\Delta} = \Delta + \bar{\Delta}, \quad \tilde{\Delta}^* = \Delta - \bar{\Delta}, \quad (2.12)$$

we can rewrite Eq. (2.11) in the compact form:

$$(i\hat{p} + \Delta - \hat{\sigma}_1 V - \hat{\sigma}_3 \bar{\Delta})\hat{G}(\mathbf{r}, \mathbf{r}', p) = \delta(\mathbf{r} - \mathbf{r}'), \quad (2.13)$$

where \hat{G} is the matrix of the functions G and F involved in Eq. (2.11), $\hat{p} = \hat{\sigma}_1 p + \hat{\sigma}_2(h - \varepsilon_F)$, $\hat{\sigma}_\alpha$ are the Pauli matrices, and $p = -i\varepsilon$. We omit the unit matrices before the terms with Δ and $\delta(\mathbf{r} - \mathbf{r}')$. The functions G and F can be written as traces of \hat{G} in the following way: ²

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}', \varepsilon) &= \frac{1}{2} \text{Sp}\{(\hat{\sigma}_1 - i\hat{\sigma}_2)\hat{G}(\mathbf{r}, \mathbf{r}', p)\}, \\ F(\mathbf{r}, \mathbf{r}', \varepsilon) &= \frac{1}{2} \text{Sp}\{(1 + \hat{\sigma}_3)\hat{G}(\mathbf{r}, \mathbf{r}', p)\}. \end{aligned} \quad (2.14)$$

Therefore, the equation for $\tilde{\Delta}(\mathbf{r})$ is

$$\tilde{\Delta}(\mathbf{r}) = g \int_{C'} \frac{dp}{4\pi} \text{Sp}\{(1 + \hat{\sigma}_3)\hat{G}(\mathbf{r}, \mathbf{r}, p)\}, \quad (2.15)$$

and the one-particle density matrix of the system is given by the expression

$$\rho(\mathbf{r}) = \int_{C'} \frac{dp}{4\pi} \text{Sp}\{(\hat{\sigma}_1 - i\hat{\sigma}_2)\hat{G}(\mathbf{r}, \mathbf{r}, p)\}, \quad (2.16)$$

where the contour C' is obtained from C by the 90° rotation.

²We use the symbol tr in the space of single-particle states of the Hamiltonian h , the symbol Sp in the spinor space, and Tr in the combined space.

2.4 Perturbation theory

We apply the Green's function method to calculate the parameter \mathcal{B} . As follows from Eqs. (1.5) and (1.7), this requires the perturbation theory of third order in the cranking term V . According to Ref. [13], a considerable computational effort is needed to mold the result into a tractable form. The matrix representation of the Gor'kov equations allows us to elaborate on the elegant form of the perturbation theory which considerably simplifies the calculations.

We now proceed to treat Eq. (2.13) by the method of successive approximation. Considering V as a weak perturbation, we expand the Green's function and the self-consistent quantities $\tilde{\Delta}$ and ε_F in the powers of a small parameter:

$$\begin{aligned}\hat{G} &= \hat{G}_0 + \hat{G}_1 + \hat{G}_2 + \hat{G}_3 + \dots, \\ \tilde{\Delta} &= \Delta^{(0)} + \bar{\Delta}^{(1)} + \Delta^{(2)} + \bar{\Delta}^{(3)} + \dots, \quad \varepsilon_F = \varepsilon_F^{(0)} + \varepsilon_F^{(2)} + \dots\end{aligned}\quad (2.17)$$

The n th order corrections to the last two quantities are determined by Eqs. (2.15) and (2.5), respectively. Since $V = -V^*$, the corrections of odd order to $\tilde{\Delta}$ are purely imaginary and those of even order are real. The effect of the second order correction to ε_F gives a negligible small contribution in the second inertial parameter [13]. Thus, we will use the zero-order approximation for this quantity.

It is natural to work in the basis of the eigenfunctions of the Hamiltonian (2.1) without the cranking term,

$$(h - \varepsilon_F)\varphi_\nu(\mathbf{r}) = p_\nu\varphi_\nu(\mathbf{r}), \quad (2.18)$$

where p_ν is the energy ε_ν of the single-particle state ν relative to the Fermi energy, $p_\nu = \varepsilon_\nu - \varepsilon_F$. In this basis, the n th correction to the Green's function has the form

$$\hat{G}_n(\mathbf{r}, \mathbf{r}', p) = \sum_{\nu\nu'} \hat{G}_{\nu\nu'}^{(n)}(p) \varphi_\nu(\mathbf{r}) \varphi_{\nu'}^*(\mathbf{r}'). \quad (2.19)$$

First of all, we find the solution of the unperturbed equation (2.13),

$$(i\hat{p} + \Delta)\hat{G}_0(\mathbf{r}, \mathbf{r}', p) = \delta(\mathbf{r} - \mathbf{r}'), \quad (2.20)$$

with the constant pairing gap $\Delta^{(0)} = \Delta$. Substituting Eq. (2.19) into (2.20), one finds

$$\hat{G}_{\nu\nu'}^{(0)}(\mathbf{p}_\nu) = -\frac{i\hat{p}_\nu - \Delta}{\mathbf{p}_\nu^2 + \Delta^2} \delta_{\nu\nu'}, \quad (2.21)$$

where $\mathbf{p}_\nu(p, p_\nu)$ is the two-dimensional vector and $\hat{p}_\nu = \hat{\sigma}_1 p + \hat{\sigma}_2 p_\nu$. The gap equation (2.15) takes the simple form

$$1 = g \sum_\nu \frac{1}{2E_\nu} |\varphi_\nu(\mathbf{r})|^2, \quad E_\nu = \sqrt{p_\nu^2 + \Delta^2}. \quad (2.22)$$

The equation has the solution $\Delta = \text{const}$ for the self-consistent potential with a flat bottom. In obtaining Eq. (2.22) as well as in subsequent calculations, it is essential to compute the traces of the products of the Pauli matrices. We can readily see that the trace of an odd number of matrices $\hat{\sigma}_1$ and $\hat{\sigma}_2$ vanishes and that of an even number is given by the expressions

$$\frac{1}{2} \text{Sp}(\hat{\sigma}_\alpha \hat{\sigma}_\beta) = \delta_{\alpha\beta}, \quad \frac{1}{2} \text{Sp}(\hat{\sigma}_\alpha \hat{\sigma}_\beta \hat{\sigma}_\gamma \hat{\sigma}_\delta) = \delta_{\alpha\beta} \delta_{\gamma\delta} - \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta}, \quad \dots, \quad (2.23)$$

where $\hat{\varepsilon} = i\hat{\sigma}_2$ is a fully antisymmetric matrix.

In the first order, Eq. (2.13) involves the two perturbing terms V and $\bar{\Delta}^{(1)}$:

$$(i\hat{p} + \Delta)\hat{G}_1(\mathbf{r}, \mathbf{r}', p) = [\hat{\sigma}_1 V + \hat{\sigma}_3 \bar{\Delta}^{(1)}]\hat{G}_0(\mathbf{r}, \mathbf{r}', p). \quad (2.24)$$

The solution of this equation is obvious:

$$\hat{G}_1(\mathbf{r}, \mathbf{r}', p) = \int \hat{G}_0(\mathbf{r}, \mathbf{q}, p) \hat{W}(\mathbf{q}) \hat{G}_0(\mathbf{q}, \mathbf{r}', p) d\mathbf{q}, \quad (2.25)$$

where the operator

$$\hat{W} = \hat{\sigma}_1 V + \hat{\sigma}_3 \bar{\Delta}^{(1)} \quad (2.26)$$

is introduced to write the corrections to the unperturbed Green's function in the simple symbolic form:

$$\begin{aligned} \hat{G}_1 &= \hat{G}_0 \hat{W} \hat{G}_0, & \hat{G}_2 &= \hat{G}_0 \hat{W} \hat{G}_0 \hat{W} \hat{G}_0 - \hat{G}_0 \Delta^{(2)} \hat{G}_0, \\ \hat{G}_3 &= \hat{G}_0 \hat{W} \hat{G}_0 \hat{W} \hat{G}_0 \hat{W} \hat{G}_0 - \hat{G}_0 \hat{W} \hat{G}_0 \Delta^{(2)} \hat{G}_0 - \hat{G}_0 \Delta^{(2)} \hat{G}_0 \hat{W} \hat{G}_0 + \hat{G}_0 \hat{\sigma}_3 \bar{\Delta}^{(3)} \hat{G}_0. \end{aligned} \quad (2.27)$$

Here the integration over intermediate coordinates \mathbf{q} is implied. Using these formulas, one can prove by straightforward calculations the following identities:

$$\begin{aligned} \int_{C'} \frac{dp}{2\pi} \text{Sp}\{\hat{G}_{2i+1}(\mathbf{r}, \mathbf{r}', p)\} &= 0, & \int_{C'} \frac{dp}{2\pi} \text{Sp}\{\hat{\sigma}_1 \hat{G}_{2i}(\mathbf{r}, \mathbf{r}', p)\} &= 0, \\ \int_{C'} \frac{dp}{2\pi} \text{Sp}\{\hat{\sigma}_2 \hat{G}_{2i+1}(\mathbf{r}, \mathbf{r}', p)\} &= 0, & \int_{C'} \frac{dp}{2\pi} \text{Sp}\{\hat{\sigma}_3 \hat{G}_{2i}(\mathbf{r}, \mathbf{r}', p)\} &= 0. \end{aligned} \quad (2.28)$$

In order to find the self-consistent solution, we have to show how $\tilde{\Delta}^{(n)}$ is obtained from \hat{G}_n . We will consider the unperturbed pairing gap Δ as a parameter of the theory. This allows us to eliminate the interaction constant g . Multiplying the zero-order equation (2.22) by $\tilde{\Delta}^{(n)}(\mathbf{r})$ we write the result in the symmetric form:

$$\Delta \tilde{\Delta}^{(n)}(\mathbf{r}) = g \int_{C'} \frac{dp}{8\pi} \text{Sp}[\hat{G}_0(\mathbf{r}, \mathbf{r}, p), \tilde{\Delta}^{(n)}(\mathbf{r})]_+, \quad (2.29)$$

where $[\dots]_+$ is the anticommutator of corresponding operators. With this ansatz, the integral equation for even order corrections is given by:

$$\int_{C'} \frac{dp}{2\pi} \text{Sp}\{2\Delta \hat{G}_{2i}(\mathbf{r}, \mathbf{r}, p) - [\hat{G}_0(\mathbf{r}, \mathbf{r}, p), \Delta^{(2i)}(\mathbf{r})]_+\} = 0, \quad (2.30)$$

while that for odd order ones has the form:

$$\int_{C'} \frac{dp}{2\pi} \text{Sp}\{2\Delta \hat{\sigma}_3 \hat{G}_{2i+1}(\mathbf{r}, \mathbf{r}, p) - [\hat{G}_0(\mathbf{r}, \mathbf{r}, p), \bar{\Delta}^{(2i+1)}(\mathbf{r})]_+\} = 0. \quad (2.31)$$

Similarly, eliminating g from the equations for $\bar{\Delta}^{(n)}$ and $\Delta^{(n-1)}$ (n is odd) yields

$$\int_{C'} \frac{dp}{2\pi} \text{Sp}\{\bar{\Delta}^{(n)}(\mathbf{r}) \hat{G}_{n-1}(\mathbf{r}, \mathbf{r}, p) - \Delta^{(n-1)}(\mathbf{r}) \hat{\sigma}_3 \hat{G}_n(\mathbf{r}, \mathbf{r}, p)\} = 0. \quad (2.32)$$

In the same manner, with additional integration over $d\mathbf{r}$, we can derive

$$\int_{C'} \frac{dp}{2\pi} \text{Tr} \{ \bar{\Delta}^{(n)} \hat{\sigma}_3 \hat{G}_{n-2}(p) - \bar{\Delta}^{(n-2)} \hat{\sigma}_3 \hat{G}_n(p) \} = 0. \quad (2.33)$$

Here Tr refers to the spinor and to the single-particle spaces simultaneously.

The solution of the HFB equations presented in the form or the successive approximations by the formulas (2.21), (2.22), (2.27), (2.30), (2.31) takes into consideration nonuniform pairing correlations induced by rotation. The nonuniform pairing originates in higher orders of the perturbation theory, while the nonrotating system is approximated by the constant pairing field. In spite of an apparent eclecticism, our solution does not violate the current density conservation, $\text{div } \mathbf{j} = 0$. This is obvious in the zero-order approximation. In the even orders of the perturbation theory $\text{div } \mathbf{j}^{(2i)}$ vanishes due to the first and four identities of (2.28). The odd corrections to the quantity (2.10) can be transformed into the expression

$$\text{div } \mathbf{j}^{(2i+1)}(\mathbf{r}) = \sum_{l=0}^i \int_{C'} \frac{dp}{2\pi} \text{Sp} \{ \bar{\Delta}^{(2l+1)}(\mathbf{r}) \hat{G}_{2i-2l}(\mathbf{r}, \mathbf{r}, p) - \Delta^{(2l)} \hat{\sigma}_3 \hat{G}_{2i-2l+1}(\mathbf{r}, \mathbf{r}, p) \}, \quad (2.34)$$

which vanishes due to the equation (2.32). Thus, we have found the consistent solution, which is certainly more general than initially supposed. It can be used for an arbitrary single-particle potential without danger of coming to contradiction (see also Ref. [21]). We will apply this solution to calculation of the second inertial parameter.

Finally, the following identities are useful in calculations:

$$2\Delta \int_{C'} \frac{dp}{2\pi} \text{Tr} \{ \Delta^{(2i)} \hat{G}_{2i}(p) \} = \int_{C'} \frac{dp}{2\pi} \text{Tr} \{ \Delta^{(2i)} [\Delta^{(2i)}, \hat{G}_0(p)]_+ \}, \quad (2.35)$$

$$2\Delta \int_{C'} \frac{dp}{2\pi} \text{Tr} \{ \bar{\Delta}^{(2i+1)} \hat{\sigma}_3 \hat{G}_{2i+1}(p) \} = \int_{C'} \frac{dp}{2\pi} \text{Tr} \{ \bar{\Delta}^{(2i+1)} [\bar{\Delta}^{(2i+1)}, \hat{G}_0(p)]_+ \}. \quad (2.36)$$

These identities are obtained by multiplying Eq. (2.30) and (2.31) by $\Delta^{(2i)}(\mathbf{r})$ and $\bar{\Delta}^{(2i+1)}(\mathbf{r})$, respectively, and integrating over $d\mathbf{r}$.

3 Calculation of the second inertial parameter

3.1 General expression

As follows from Eq. (1.7), the derivation of the parameter \mathcal{B} is reduced to the calculation of β . This latter quantity is convenient to deal with. The third-order correction to the density matrix can be obtained from Eqs. (2.16) and (2.27). If we take into account the third equation from (2.28), substitute

$$\hat{\sigma}_1 \ell_x = -\frac{1}{\omega} (W - \hat{\sigma}_3 \Delta^{(1)}), \quad (3.1)$$

and use the identity (2.33) with $n = 3$, we get after some simple algebraic calculations

$$\beta_s = \frac{1}{\omega^3} \text{tr} \{ \ell_x \rho^{(3)} \} = -\frac{1}{\omega^4} \int_{C'} \frac{dp}{4\pi} \text{Tr} \{ \hat{W} \hat{G}_3(p) \} + \frac{1}{\omega^4} \int_{C'} \frac{dp}{4\pi} \text{Tr} \{ \Delta^{(3)} \hat{\sigma}_3 \hat{G}_1(p) \}, \quad (3.2)$$

where the subscript s means that the relevant quantity refers to the superfluid state. Here and below we will use $\Delta^{(2i+1)}$ ($i = 0, 1$) instead of $\bar{\Delta}^{(2i+1)}$. Fortunately, the terms with $\Delta^{(3)}$ are eliminated from (3.2) after inserting the expression for \hat{G}_3 (2.27). The resulting formula involves the corrections $\Delta^{(1)}$ and $\Delta^{(2)}$ only. It is convenient to transform the terms with $\Delta^{(2)}$ into a quadratic form in this quantity. Referring to the definition of the function \hat{G}_2 from Eqs. (2.27), we find that the term involving $\Delta^{(2)}$ becomes

$$\begin{aligned} & \int_{C'} \frac{dp}{2\pi} \text{Tr} \{ \Delta^{(2)} \hat{G}_0(p) \hat{W} \hat{G}_0(p) \hat{W} \hat{G}_0(p) \} \\ &= \int_{C'} \frac{dp}{2\pi} \text{Tr} \{ \Delta^{(2)} \hat{G}_2(p) \} - \int_{C'} \frac{dp}{2\pi} \text{Tr} \{ \Delta^{(2)} \hat{G}_0(p) \Delta^{(2)} \hat{G}_0(p) \}. \end{aligned} \quad (3.3)$$

With the help of the identity (2.35) it is easy to show that the first term on the right also transforms into a bilinear form of $\Delta^{(2)}$. Combining this result with (3.2) we obtain the final expression for the parameter β_s :

$$\begin{aligned} \beta_s &= -\frac{1}{\omega^4} \int_{C'} \frac{dp}{4\pi} \text{Tr} \{ \hat{W} \hat{G}_0(p) \hat{W} \hat{G}_0(p) \hat{W} \hat{G}_0(p) \hat{W} \hat{G}_0(p) \} \\ &+ \frac{1}{\omega^4 \Delta} \int_{C'} \frac{dp}{4\pi} \text{Tr} \{ 2\Delta \Delta^{(2)} \hat{G}_0(p) \Delta^{(2)} \hat{G}_0(p) + \Delta^{(2)} [\hat{G}_0(p), \Delta^{(2)}]_+ \}. \end{aligned} \quad (3.4)$$

This is an exact formula for the calculation of the second inertial parameter. The first term describes the joint effect of the Coriolis force and the nonuniform pairing field $\Delta^{(1)}(\mathbf{r})$ on independent quasiparticle motion. In the limit of the monopole pairing interaction, which corresponds to the uniform pairing field ($\Delta^{(1)} = 0$), this agrees with the term \mathcal{B}_c found by Marshalek [13]. The second term arises only due to the modification of the pairing. In the limit of uniform pairing this term describes the Coriolis antipairing effect (see Appendix A).

3.2 Semiclassical approximation

To proceed further we should find the first and second order corrections to the pairing field. As shown in Appendix A, the solutions of the corresponding integral equations are

$$\Delta^{(1)}(\mathbf{r}) = -\frac{i\hbar\omega}{2\Delta} D_1 \dot{\ell}_x, \quad \Delta^{(2)}(\mathbf{r}) = \frac{\hbar^2 \omega^2}{4\Delta^3} D_2 \dot{\ell}_x^2, \quad (3.5)$$

where D_1 and D_2 are the amplitudes, which are found in a self-consistent way. In order to learn more about the nonuniform pairing, we suppose that the self-consistent potential of a deformed nucleus is of the form

$$U(\rho) = U \left(\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} \right), \quad (3.6)$$

where a and b are the half-axes of a nuclear spheroid. Then we obtain

$$\dot{\ell}_x = y \frac{\partial U}{\partial z} - z \frac{\partial U}{\partial y} = \frac{b^2 - a^2}{a^2 b^2} y z U'(\rho). \quad (3.7)$$

Thus, the rotation induced pairing field is a function of the space coordinates only. In the first order in rotation, the nonuniform pairing field is proportional to the spherical harmonic $Y_{2\pm 1}$. This was a motive for introducing the quadrupole pairing (see [27] and references therein). The second correction $\Delta^{(2)}$ shows, however, that higher multipoles are also involved in nonuniform pairing.

Using the expression for $\Delta^{(1)}$ and the obvious formula for matrix elements $\hbar \dot{\ell}_{\nu\nu'}^x = i(p_\nu - p_{\nu'})\ell_{\nu\nu'}^x$, we can represent each of 16 sums in the first term of Eq. (3.4) in the general form

$$\sum \ell_{12}^x \ell_{23}^x \ell_{34}^x \ell_{41}^x \int_{C'} \frac{dp}{2\pi} \frac{\mathcal{Q}_4(p; p_1, p_2, p_3, p_4)}{(\mathbf{p}_1^2 + \Delta^2)(\mathbf{p}_2^2 + \Delta^2)(\mathbf{p}_3^2 + \Delta^2)(\mathbf{p}_4^2 + \Delta^2)}, \quad (3.8)$$

where the summation indices 1,2,3,4 refer to the single-particle states ν with the energy p_ν of the Schrödinger equation (2.18) and \mathcal{Q}_4 is a polynomial of the fourth power in (p, p_ν) and D_1 , which is derived from calculating the trace of the product of the Pauli matrixes and factors $(i\hat{p}_\nu - \Delta)$.

To evaluate this sum we use the method proposed by Migdal [21]. Let us note first that at the fixed state 1 the indexes 2, 3, and 4 take only few values permitted by the selection rules for the matrix elements of the operator ℓ_x . After integration over dp we obtain the function of the variables p_1, \dots, p_4 . When considered as the function of the variable p_1 , it has for fixed differences $p_1 - p_2$, $p_1 - p_3$, and $p_1 - p_4$ a sharp maximum at the Fermi surface with an approximate width Δ . Since the average level spacing for ND nuclei, $\delta\varepsilon \sim \varepsilon_F/A$, is very small compared to $\Delta \sim \varepsilon_F/A^{2/3}$, a large number ($\sim A^{1/3}$) of levels fall within the interval Δ . For this reason we can make with semiclassical accuracy the following substitution in the sum (3.8):

$$\begin{aligned} & \int_{C'} \frac{dp}{2\pi} \frac{\mathcal{Q}_4(p; p_1, p_2, p_3, p_4)}{(\mathbf{p}_1^2 + \Delta^2)(\mathbf{p}_2^2 + \Delta^2)(\mathbf{p}_3^2 + \Delta^2)(\mathbf{p}_4^2 + \Delta^2)} \\ & \rightarrow \delta(\varepsilon_1 - \varepsilon_F) \int \frac{d\mathbf{p}_1}{2\pi} \frac{\mathcal{Q}_4(p; p_1, p_2, p_3, p_4)}{(\mathbf{p}_1^2 + \Delta^2)(\mathbf{p}_2^2 + \Delta^2)(\mathbf{p}_3^2 + \Delta^2)(\mathbf{p}_4^2 + \Delta^2)}. \end{aligned} \quad (3.9)$$

Similarly, the second term of Eq. (3.4) can be approximated by the expression

$$\frac{1}{\omega^4} \sum |\Delta_{12}^{(2)}|^2 \int \frac{d\mathbf{p}_1}{4\pi} \frac{(p_1 - p_2)^2 + 4\Delta^2}{(\mathbf{p}_1^2 + \Delta^2)(\mathbf{p}_2^2 + \Delta^2)} \delta(\varepsilon_1 - \varepsilon_F). \quad (3.10)$$

When calculating these integrals over $d\mathbf{p}_1 = dp dp_1$, it is convenient to use the Feynman covariant integration method [28] because good convergence of the integrals allows the integration over dp_1 to be extended from $-\infty$ to ∞ . For details of calculations see Appendix B. The integral in (3.9) depends on the three independent differences $p_1 - p_2$, $p_1 - p_3$, and $p_1 - p_4$. To represent the final result of the semiclassical approximation in a symmetrical form, we introduce the six energy differences $p_{\nu\nu'} = p_\nu - p_{\nu'}$, $\nu < \nu'$. Collecting all the integrals of the first and the second terms in (3.4), we find

$$\beta_s = \frac{1}{4\Delta^2} \sum \ell_{12}^x \ell_{23}^x \ell_{34}^x \ell_{41}^x F(x_{12}, x_{23}, x_{34}, x_{41}; x_{13}, x_{24}) \delta(\varepsilon_1 - \varepsilon_F), \quad (3.11)$$

where δ function denotes that summation over the state 1 is substituted, according to the semiclassical approximation, by integration over its quantum numbers.

The function F , which depends on the six dimensionless differences $x_{\nu\nu'} = (\varepsilon_\nu - \varepsilon_{\nu'})/2\Delta$, is divided into two parts:

$$F = f(x_{12}, x_{23}, x_{34}, x_{41}; x_{13}, x_{24}) + 8D_2^2 x_{12} x_{23} x_{34} x_{41} h(x_{13}). \quad (3.12)$$

The first one is relevant for the first term of Eq. (3.4). It is convenient to represent this function in the form

$$f = -(1 + \hat{P}_1 + \hat{P}_2 + \hat{P}_3)G + (1 + \hat{P}_1)H, \quad (3.13)$$

where

$$\begin{aligned} G = & \frac{g(x_{12})}{x_{13}x_{23}x_{24}x_{41}} \left\{ (1 - D_1 x_{12}^2) \left[1 + x_{12}^2 + x_{23}x_{41} - D_1 [x_{23}^2(1 - x_{12}x_{24}) \right. \right. \\ & + x_{41}^2(1 + x_{12}x_{13})] - D_1^2 x_{23}x_{34}x_{41}(x_{23} + x_{41}) - D_1^3 x_{12}x_{23}^2 x_{34}x_{41}^2 \Big] \\ & \left. - D_1(x_{34} - D_1 x_{12}x_{23}x_{41})(x_{34} + x_{12}x_{13}x_{24} - D_1 x_{12}x_{23}x_{41}) \right\} \end{aligned} \quad (3.14)$$

and

$$H = \frac{h(x_{13})}{x_{12}x_{23}x_{34}x_{41}} \left[1 - D_1(x_{12}^2 + x_{23}^2 + x_{34}^2 + x_{41}^2) + D_1^2(x_{12}x_{41} + x_{23}x_{34})^2 \right]. \quad (3.15)$$

Here the functions

$$g(x) = \frac{\operatorname{argsh} x}{x\sqrt{1+x^2}}, \quad h(x) = (1+x^2)g(x) \quad (3.16)$$

are associated with the Migdal moment of inertia [21]. The amplitudes D_1 and D_2 of the nonuniform pairing field are determined by Eqs. (A.4) and (A.8), respectively. The operator \hat{P}_i permutes the indices ν of single-particle states in all the dimensionless differences, on which the functions G and H depend. When applied to $x_{\nu\nu'}$, we get

$$\hat{P}_i x_{\nu,\nu'} = x_{\nu+i,\nu'+i}, \quad (3.17)$$

subject to $\nu \bmod 4 = \nu$. It is easy to prove the following symmetry properties of the function F :

$$\hat{P}_1 F = \hat{P}_2 F = \hat{P}_3 F = F,$$

$$F(x_{12}, x_{41}, x_{34}, x_{23}; -x_{24}, -x_{13}) = F(x_{12}, x_{23}, x_{34}, x_{41}; x_{13}, x_{24}). \quad (3.18)$$

The above formulas give the semiclassical expression for the second inertial parameter in the superfluid phase. The solution takes into account the effect of rotation on the Cooper pairs in the gauge-invariant form. The result is expressed entirely in terms of matrix elements and corresponding energy differences providing the constant pairing gap Δ is fixed for a nonrotating nucleus. It is valid for an arbitrary nuclear mean field with a stable deformation. This allows one to study an interplay between rotation, pairing correlations, and mean field deformation in ND and SD bands.

We first estimate the quantity β_s and find the small parameter of the perturbation theory we used. To get an estimate of β_s for ND bands, we observe that the matrix element $\ell_{\nu\nu'}^x$ has the two types of transitions $\nu \rightarrow \nu'$:

(i) transitions inside the \mathcal{N} -shell (close transitions), for which energy differences are $p_{\nu\nu'} = d_1 \sim \varepsilon_F A^{-2/3}$, and the maximal value L of the matrix element $\ell_{\nu\nu'}^x$ is related to a transition between states of a j -shell;

(ii) transitions between shells with major quantum numbers \mathcal{N} and $\mathcal{N} \pm 2$ (distant transitions) with $p_{\nu\nu'} = d_2 \sim \varepsilon_F A^{-1/3}$ and $\ell_{\nu\nu'}^x \sim LA^{-1/3}$.

For the state 1, there are the three groups of terms in the sum (3.11), which are classified according to different combinations of the close and distant transitions in the product of the four matrix elements $\ell_{\nu\nu'}^x$. Those involving four close transitions have all the dimensionless differences $x_{\nu\nu'} \sim 1$ and consequently $F \sim 1$.³ Thus, the contribution of these terms to the sum (3.11) is of the order L^4 . For terms with four distant transitions ($x_{\nu\nu'} \sim A^{1/3}$), we have $F \sim A^{4/3}$. However, this large factor is compensated by the product of small matrix elements $\ell_{\nu\nu'}^x$. The same compensation takes place in the remaining terms with two close and two distant transitions, for which $F \sim A^{2/3}$. Therefore, the contributions of all terms in the sum (3.11) are of the same order of magnitude L^4 , and we can make the following estimation:

$$\beta_s \sim \frac{1}{4\Delta^2} \sum \ell_{12}^x \ell_{23}^x \ell_{34}^x \ell_{41}^x \delta(\varepsilon_1 - \varepsilon_F) = \frac{1}{4\Delta^2} \sum_1 (\ell_x^4)_{11} \delta(\varepsilon_1 - \varepsilon_F). \quad (3.19)$$

Calculation of the last sum within the Tomas-Fermi approximation gives

$$\beta_s \sim \frac{3M}{20\Delta^2} \int n(\mathbf{r}) p_F^2(\mathbf{r}) (y^2 + z^2)^2 d\mathbf{r}. \quad (3.20)$$

In this calculation we used the procedure described in Ref. [21], which includes averaging over the direction of the nucleon impulse and an utilization of the ansatz

$$\sum_1 \varphi_1^*(\mathbf{r}) \varphi_1(\mathbf{r}) \delta(\varepsilon_1 - \varepsilon_F) = \frac{3M}{p_F^2(\mathbf{r})} n(\mathbf{r}), \quad (3.21)$$

where $n(\mathbf{r}) = Cp_F^3(\mathbf{r})$ ($C = \text{const}$) is the nucleon density and $p_F(\mathbf{r}) = \sqrt{2M[\varepsilon_F - U(\mathbf{r})]}$. Comparing (3.20) with the rigid-body moment of inertia

$$\mathfrak{I}_{\text{rig}} = \int n(\mathbf{r}) (y^2 + z^2) d\mathbf{r}, \quad (3.22)$$

we obtain $\beta_s \sim \mathfrak{I}_{\text{rig}} (p_F R / \Delta)^2 \sim \mathfrak{I}_{\text{rig}} (\hbar j_F / \Delta)^2$, where R is the mean square radius of a nucleus, p_F and $j_F \sim A^{1/3}$ are the mean impulse and the mean angular momentum of a nucleon on the Fermi surface. Thus, the parameter β_s has the order of magnitude $\hbar^4 A^{11/3} / \varepsilon_F^3$. With these results we can get from (1.4) the perturbation parameter, $\beta_s \omega^2 / \alpha \sim (\hbar \omega j_F / \Delta)^2$. An application of the perturbation theory implies that this value is small, i. e., the Coriolis interaction is smaller than a two-quasiparticle excitation energy. One can say that the perturbation theory is valid for adiabatic rotation. It is clear from Eq. (1.7) that $\mathcal{B}_s \sim \varepsilon_F A^{-3}$ and $\mathcal{B}_s / \mathcal{A} \sim A^{-4/3}$. The above estimations refer to ND nuclei in the ground state where pairing correlations are reasonably strong, $\Delta \sim \varepsilon_F A^{-2/3}$.

4 The model of anisotropic oscillator potential

In order to obtain quantitative results, we model the real self-consistent nuclear field as the axially deformed oscillator potential with the frequencies ω_z along the symmetry axis and ω_x in the perpendicular plane:

$$U_{\text{osc}}(\mathbf{r}) = \frac{M}{2} [\omega_x^2 (x^2 + y^2) + \omega_z^2 z^2]. \quad (4.1)$$

³The necessary estimation for the amplitudes of the uniform pairing field, $D_1 \sim D_2 \sim [\ln(d_2/\Delta)]^{-1} \sim 1$, can be obtained from Eqs. (A.4) and (A.8).

The use of this simplified model is justified by a possibility of deriving an exact analytical expression for the parameter β_s . It is known also that the model reproduces the experimental values of the rotational constants \mathcal{A} and \mathcal{B} for ND bands. Therefore we can expect the model to provide an useful insight in the rotational regime at superdeformation.

In an anisotropic oscillator potential the matrix element $\ell_{\nu\nu'}^x$ is non-zero for four transitions. The two close transitions have the energy differences $d_1 = \pm\hbar(\omega_x - \omega_z)$, whereas the distant ones have $d_2 = \pm\hbar(\omega_x + \omega_z)$. The corresponding dimensionless parameters are

$$\nu_{1,2} = \frac{\hbar(\omega_x \mp \omega_z)}{2\Delta} = \frac{k \mp 1}{2\xi k^{2/3}}, \quad \xi = \frac{\Delta}{\hbar\omega_0}, \quad (4.2)$$

where $\hbar\omega_0 = 41A^{-1/3}$ MeV. Hereafter we use the axis or frequency ratio $k = b/a = \omega_x/\omega_z$ and the volume conservation condition $\omega_x^2\omega_z = \omega_0^3$. It is also convenient to substitute the operator ℓ_x with its time derivative

$$\dot{\ell}_x = M(\omega_x^2 - \omega_z^2)yz, \quad (4.3)$$

which has the same selection rules.

For the fixed state 1, the sum (3.11) involves 6 terms with the four close transitions, 6 terms with the four distant transitions, and 24 terms with the two close and two distant transitions. The products of four matrix elements are equal with the semiclassical accuracy for all 36 terms of the sum, $\dot{\ell}_{12}^x \dot{\ell}_{23}^x \dot{\ell}_{34}^x \dot{\ell}_{41}^x \approx \frac{1}{36}(\dot{\ell}_x^4)_{11}$. Summation of all these terms gives

$$\beta_s = \frac{\hbar^4 \Phi_n(\nu_1, \nu_2)}{18(2\Delta)^6 \nu_1^4} \sum_1 (\dot{\ell}_x^4)_{11} \delta(\varepsilon_1 - \varepsilon_F). \quad (4.4)$$

The function Φ_n is the sum of the functions F corresponding to all 36 combinations of the close and distant transitions. The sum can be simplified by using the symmetry properties (3.18). It is convenient to represent the resulting function in the form

$$\Phi_n(\nu_1, \nu_2) = \mathcal{F}(\nu_1, \nu_2) + 2D_2^2 \mathcal{H}(\nu_1, \nu_2), \quad (4.5)$$

where

$$\begin{aligned} \mathcal{F}(\nu_1, \nu_2) = & f(\nu_1, -\nu_1, \nu_1, -\nu_1; 0, 0) + 2f(\nu_1, \nu_1, -\nu_1, -\nu_1; 2\nu_1, 0) \\ & + 2(\nu_1/\nu_2)^2 [2f(\nu_1, \nu_2, -\nu_2, -\nu_1; \nu_1 + \nu_2, 0) + 2f(\nu_1, -\nu_2, \nu_2, -\nu_1; \nu_1 - \nu_2, 0) \\ & + f(\nu_1, \nu_2, -\nu_1, -\nu_2; \nu_1 + \nu_2, -\nu_1 + \nu_2) + f(\nu_1, -\nu_2, -\nu_1, \nu_2; \nu_1 - \nu_2, -\nu_1 - \nu_2)] \\ & + (\nu_1/\nu_2)^4 [f(\nu_2, -\nu_2, \nu_2, -\nu_2; 0, 0) + 2f(\nu_2, \nu_2, -\nu_2, -\nu_2; 2\nu_2, 0)], \end{aligned} \quad (4.6)$$

and the amplitudes D_1 , D_2 , and the functions \mathcal{H} are determined by Eqs. (A.5), (A.11), and (A.12), respectively. The two terms in (4.5) describe the two distinct effects of the Coriolis force: the rotation-quasiparticle interaction and the modification of pairing.

In the Tomas-Fermi approximation, we have according to the ansatz (3.21)

$$\sum_1 (\dot{\ell}_x^4)_{11} \delta(\varepsilon_1 - \varepsilon_F) = 3CM \int \dot{\ell}_x^4(\mathbf{r}) \sqrt{2M[\varepsilon_F - U_{\text{osc}}(\mathbf{r})]} d\mathbf{r}. \quad (4.7)$$

Combining the result of integration with the expressions for the rigid-body moment of inertia (3.22) and the mean level density near the Fermi surface

$$\rho_F = \int d\mathbf{r} \sum_1 \varphi_1^*(\mathbf{r}) \varphi_1(\mathbf{r}) \delta(\varepsilon_1 - \varepsilon_F) = 3CM \int \sqrt{2M[\varepsilon_F - U_{\text{osc}}(\mathbf{r})]} d\mathbf{r}, \quad (4.8)$$

we obtain

$$\sum_1 (\dot{\ell}_x^4)_{11} \delta(\varepsilon_1 - \varepsilon_F) = \frac{18(\omega_x - \omega_z)^4 (1+k)^4}{5\rho_F (1+k^2)^2} \mathfrak{S}_{\text{rig}}^2. \quad (4.9)$$

In the case of a normal deformation, the quantity k is close to unity and consequently $\nu_2 \gg \nu_1$. Inserting (4.9) into (4.4) and using the above approximations we get for ND bands

$$\beta_s(\text{ND}) = \frac{\mathfrak{S}_{\text{rig}}^2 \Phi_{nc}(\nu_1, \nu_2)}{5\rho_F \Delta^2}. \quad (4.10)$$

The function Φ_{nc} approximates Φ_n in "the close transition limit":

$$\Phi_{nc}(\nu_1, \nu_2) = \mathcal{F}_c(\nu_1, \nu_2) + 2D_2^2 \mathcal{H}_c(\nu_2), \quad (4.11)$$

where

$$\mathcal{F}_c(\nu_1, \nu_2) = f(\nu_1, -\nu_1, \nu_1, -\nu_1; 0, 0) + 2f(\nu_1, \nu_1, -\nu_1, -\nu_1; 2\nu_1, 0). \quad (4.12)$$

Φ_n depends on ν_2 only through the amplitudes D_1 , D_2 , and the function

$$\mathcal{H}_c(\nu_2) = 8 + 8 \ln 2\nu_2 + \ln 4\nu_2. \quad (4.13)$$

The level density near the Fermi surface can be obtained by combining (4.8) with the number of nucleons $A = \int n(\mathbf{r}) d\mathbf{r}$. The result, $\rho_F = 3A/\varepsilon_F$, depends on the Fermi energy which is found from the volume conservation condition $a^2 b = R^3$. The expression (4.10) has been obtained in Ref. [11].

Figure 2a shows that Φ_{nc} approximates the exact function Φ_n very well. It shows also that the contribution of the rotation-quasiparticle interaction is small compared to that of the pairing modification. This result is explained by the interference of the two effects: the Coriolis force and the nonuniform pairing field $\Delta^{(1)}$. Neglecting the latter results in compatible contributions of the two terms of Eq. (4.11) as it is seen from Fig. 2b. This result is consistent with the Marshalek calculations [13].

For superdeformed nuclei the parameters ν_1 and ν_2 are both large. Thus, we should expect decrease of β_s . It is convenient to rewrite Eq. (4.4) by introducing, according to (4.2), the new parameters ξ and k instead of ν_1 and ν_2 . We express the sum (4.7) in terms of the rigid-body moment of inertia and the number of nucleons A :

$$\sum_1 (\dot{\ell}_x^4)_{11} \delta(\varepsilon_1 - \varepsilon_F) = \frac{24(\omega_x - \omega_z)^4 k^{2/3} (1+k)^4}{5(1+k^2)^3 A^2} \omega_0^2 \mathfrak{S}_{\text{rig}}^3. \quad (4.14)$$

We have now:

$$\beta_s(\text{SD}) = \frac{k^{2/3} (1+k)^4}{15\hbar^2 (1+k^2)^3 A^2} \mathfrak{S}_{\text{rig}}^3 \Phi(\xi, k), \quad \xi^2 \Phi(\xi, k) = \Phi_n(\nu_1, \nu_2). \quad (4.15)$$

The function Φ , along with its limiting case of the uniform pairing, is shown in Fig. 3. It can be seen that nonuniform pairing reduces $\beta_s(\text{SD})$ even more than $\beta_s(\text{ND})$.

Taking $k = 2$ and $\Delta = 0.5 \text{ MeV}$ ($\xi = 0.065$) as the representative parameters for SD bands, we find from Fig. 3 that $\Phi \sim 1$. This yields the following estimation: $\beta_s(\text{SD}) \sim \hbar^4 (A/\varepsilon_F)^3 \sim \beta_s(\text{ND}) A^{-2/3}$ and $\mathcal{B}/\mathcal{A} \sim A^{-2}$. The last estimation is correct for the SD bands in the $A \sim 190$ mass region where $\mathcal{B}/\mathcal{A} \sim 10^{-5}$. Yet it overestimates the experimental value for $^{84}\text{Zr}(1)$ ($\sim 10^{-5}$) and $^{144}\text{Gd}(1)$, $^{152}\text{Dy}(1)$ ($\sim 10^{-6}$). The later two bands have the smallest value of this ratio among all SD mass regions. Thus, a high deformation and nonuniform pairing do not solve the problem of the SD band regularity.

4.1 Limiting cases

The limiting cases most interesting to us are: strong pairing, uncorrelated nucleons, and extremely large deformations. Reference [11] considers the limit of small deformations.

For a very strong pairing ($\Delta \gg \hbar\omega_0$), the size of the Cooper pair $R\hbar\omega_0/\Delta$ becomes much less than the nuclear radius R . Rotation of such nucleus is described by the hydrodynamic equations of the ideal liquid [9], according to which the second inertial parameter vanishes. In the following analysis, the key aspect is the nonuniform pairing. For strong pairing, the quantities $\Delta^{(1)}$ and $\Delta^{(2)}$ are proportional to Δ because, as follows from Eqs. (A.4) and (A.8), $D_1 \sim \Delta^2$ and $D_2 \sim \Delta^4$. Therefore this limit is instructive since it allows to check the solution of the integral equation for $\Delta^{(2)}$.⁴

In the strong pairing limit the parameters ν_1 and ν_2 are small. It is possible to simplify the function F by expanding $g(x_{\nu\nu'})$ and $h(x_{\nu\nu'})$ in power series of $x_{\nu\nu'}$, and then approximate with a necessary accuracy by

$$F = \mathcal{P}_2 + D_1\mathcal{P}_4 + D_1^2\mathcal{P}_4 + D_1^3\mathcal{P}_6 + D_1^4\mathcal{P}_8, \quad (4.16)$$

where \mathcal{P}_n is the polynomial of the n th power in $x_{\nu\nu'}$. With this function, performing a calculation similar to the one we used to obtain Φ_n , we find the limiting value

$$\lim_{\Delta \rightarrow \infty} \Phi_n(\nu_1, \nu_2) = -\frac{64}{3} \left(\frac{\omega_x - \omega_z}{\omega_x + \omega_z} \right)^2. \quad (4.17)$$

Combining this result with Eqs. (4.4) and (4.14) gives

$$\beta_s \sim -\frac{\mathfrak{S}_{\text{rig}}^3}{(\hbar A)^2} \left(\frac{b^2 - a^2}{b^2 + a^2} \right)^2 \left(\frac{\hbar\omega_0}{\Delta} \right)^2. \quad (4.18)$$

Thus, the parameter β_s vanishes in the hydrodynamic limit.

The rotation of a very elongated nucleus with $k = b/a \gg 1$ exhibits some interesting physics. For this limit, the parameters ν_1 and ν_2 are approximately equal

$$\nu_1 = \nu \pm \delta\nu, \quad \nu = \frac{\omega_x}{2\Delta}, \quad \delta\nu = \frac{\omega_z}{2\Delta}, \quad (4.19)$$

where $\nu \gg 1$ and $\delta\nu/\nu = a/b \ll 1$. The nonuniform pairing is also important in this case because the small amplitude $D_1 \approx 1/\nu^2$ is compensated by the large value of ν^2 . As a result, the kinematic moment of inertia, which is the sum of the standard cranking-model term and the Migdal one, is close to the rigid body value:

$$\mathfrak{S}^{(1)} = \mathfrak{S}_{\text{rig}} \left[1 - \left(\frac{a}{b} \right)^2 \frac{10}{\nu^2} \ln 2\nu \right]. \quad (4.20)$$

For the first approximation $\nu_1 = \nu_2$, the function (4.5) vanishes, $\Phi_n(\nu, \nu) = 0$. The next term of its expansion in $\delta\nu$ gives the estimation of the second inertial parameter $\beta_s \sim (a/b)^2$. We can say that a strongly elongated nucleus in the superfluid phase has the rotational regime which is close to the rigid-body rotation. The deviation are of the order $(a/b)^2$. The physical interpretation of this phenomenon is straightforward: all nucleons of a needle

⁴The solution for $\Delta^{(1)}$ has been verified in Ref. [21] by obtaining the hydrodynamic moment of inertia.

shaped nucleus with exclusion of the small sphere of the radius a in its center are involved in rotational motion.

Let us now consider the normal phase. The right-hand side of Eq.(3.11) vanishes in the limiting case $\Delta = 0$. This result is an artifact of the semiclassical approximation used in deriving the expression (3.11). The correct formula obtained from Eq. (3.4) with the limiting values of the Bogolubov amplitudes ($u_\nu = 0$, $v_\nu = 1$ for $\rho_\nu = 1$ and $u_\nu = 1$, $v_\nu = 0$ for $\rho_\nu = 0$, where ρ_ν is the nucleon occupation numbers) is

$$\beta_n^{(\text{sp})} = - \sum \ell_{12}^x \ell_{23}^x \ell_{34}^x \ell_{41}^x \sum_{i=0}^3 \hat{P}_i \left\{ \frac{\rho_1}{(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)} \right\}. \quad (4.21)$$

This expression describes the effect of the Coriolis force on single-particle motion. It will be shown in the next section that the cancellation of the leading terms in the sum of (4.21) substantially reduces this quantity compared to β_s .

4.2 The second inertial parameter for uncorrelated nucleons

In this subsection we estimate the parameter β in the normal phase. In view of the cancellation mentioned above, we have to take into account the centrifugal stretching effect, which happens to be the same order of magnitude as (4.21). As with the superfluid phase, we will use the Green's function technique.⁵ Our starting point is the equations in the Hartree approximation⁶

$$[\varepsilon - h_\omega(\mathbf{r}) - \mathcal{V}(\mathbf{r})]G(\mathbf{r}, \mathbf{r}', \varepsilon) = \delta(\mathbf{r} - \mathbf{r}'), \quad (4.22)$$

where h_ω is the cranked single-particle Hamiltonian (2.1) with the oscillator potential (4.1) and

$$\mathcal{V}(\mathbf{r}) = \int d\mathbf{q} v_q(\mathbf{r}, \mathbf{q}) \oint_C \frac{d\varepsilon}{2\pi i} G(\mathbf{q}, \mathbf{q}, \varepsilon) \quad (4.23)$$

is the self-consistent potential. We assume that the two-body residual interaction v_q is the effective quadrupole one,

$$v_q(\mathbf{r}, \mathbf{r}_1) = -\frac{\chi}{2} \sum_{\mu} (-1)^{\mu} q_{2\mu}(\mathbf{r}'') q_{2-\mu}(\mathbf{r}_1''), \quad (4.24)$$

where the quadrupole moment $q_{2\mu}$ is defined in terms of the doubly stretched coordinates

$$\mathbf{r}'' = \mathbf{i} \frac{\omega_x}{\omega_0} x + \mathbf{j} \frac{\omega_y}{\omega_0} y + \mathbf{k} \frac{\omega_z}{\omega_0} z. \quad (4.25)$$

The interaction strength χ is determined in a self-consistent way as follows:

$$\chi = \frac{4\pi\omega_0^2 M}{5\text{tr}\{(\mathbf{r}'')^2 \rho\}}. \quad (4.26)$$

This interaction provides the full self-consistency for deformed nuclei [29].

⁵There is an alternative method based on minimization of the system energy in the rotating frame as a function of the oscillator frequencies $\omega_x, \omega_y, \omega_z$, and the rotational frequency ω under the constraint of constant volume. For the fixed occupation of single particle states, this method gives the same result as the one obtained below.

⁶This approximation is adequate for a separable two-body interaction we will use.

As usual, we proceed to treat the cranking term V with the perturbation theory by expanding the Green's function and the self-consistent potential in the series

$$G = G_0 + G_1 + G_2 + G_3 + \dots, \quad \mathcal{V} = \mathcal{V}^{(0)} + \mathcal{V}^{(1)} + \mathcal{V}^{(2)} + \mathcal{V}^{(3)} + \dots \quad (4.27)$$

The unperturbed Green's function is

$$G_0(\mathbf{r}, \mathbf{r}_1, \varepsilon) = \sum_{\nu} G_{\nu}(\varepsilon) \varphi_{\nu}(\mathbf{r}) \varphi_{\nu}^*(\mathbf{r}_1), \quad G_{\nu}(\varepsilon) = \frac{1}{\varepsilon - \varepsilon_{\nu} + i\delta(1 - 2\rho_{\nu})}, \quad (4.28)$$

with $\delta \rightarrow +0$. The occupation numbers ρ_{ν} refer to a nonrotating nucleus. We may notice that under the self-consistent condition

$$\omega_x \Sigma_x = \omega_y \Sigma_y = \omega_z \Sigma_z, \quad \Sigma_{x,y,z} = \sum_{\nu} (n_{x,y,z} + 1/2)_{\nu} \rho_{\nu} \quad (4.29)$$

(n_x, n_y, n_z are the oscillator quantum numbers) $\mathcal{V}^{(0)} = 0$. Thus, the average potential is modified by rotation only.

The odd corrections to the self-consistent potential are lacking, $\mathcal{V}^{(2i+1)} = 0$, due to the different symmetry properties of the operators $q_{2\mu}$ and ℓ_x under the time reversal. Consequently the third order correction to the Green's function is expressed as

$$G_3 = G_0 V G_0 V G_0 V G_0 + G_0 V G_0 \mathcal{V}^{(2)} G_0 + G_0 \mathcal{V}^{(2)} G_0 V G_0. \quad (4.30)$$

The first term generates the interaction of rotation with single-particle motion. It yields the quantity $\beta_n^{(\text{sp})}$ (4.21). The last two are responsible for the centrifugal-stretching effect which is described by the expression

$$\beta_n^{(\text{str})} = -\frac{2}{\omega^4} \oint_C \frac{d\varepsilon}{2\pi i} \text{tr}\{\mathcal{V}^{(2)} G_0(\varepsilon) V G_0(\varepsilon) V G_0(\varepsilon)\}, \quad (4.31)$$

where the correction to the mean potential $\mathcal{V}^{(2)}$ is obtained from the equation

$$\mathcal{V}^{(2)}(\mathbf{r}) = -\chi \sum_{\mu} (-1)^{\mu} q_{2-\mu}(\mathbf{r}) \oint_C \frac{d\varepsilon}{2\pi i} \text{tr}\{q_{2\mu} [G_0(\varepsilon) V G_0(\varepsilon) V G_0(\varepsilon) + G_0(\varepsilon) \mathcal{V}^{(2)} G_0(\varepsilon)]\}. \quad (4.32)$$

The solution of this equation has the form

$$\mathcal{V}^{(2)}(\mathbf{r}) = -\omega^2 \sum_{\mu} \frac{\chi}{1 + \chi \sigma_{\mu}} (-1)^{\mu} Q_{2\mu}^{(2)} q_{2-\mu}(\mathbf{r}), \quad (4.33)$$

where

$$\sigma_{\mu} = \sum_{1,2} |(q_{2\mu})_{12}|^2 \frac{\rho_1 - \rho_2}{\varepsilon_1 - \varepsilon_2}, \quad Q_{2\mu}^{(2)} = \oint_C \frac{d\varepsilon}{2\pi i} \text{tr}\{q_{2\mu} G_0(\varepsilon) \ell_x G_0(\varepsilon) \ell_x G_0(\varepsilon)\}. \quad (4.34)$$

The last quantity is the second correction to the nuclear quadrupole moment due to rotation. Its explicit form is

$$Q_{2\mu}^{(2)} = \sum (q_{2\mu})_{12} \ell_{23}^x \ell_{31}^x \sum_{i=0}^2 \hat{P}_i \left\{ \frac{\rho_1}{(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)} \right\}, \quad (4.35)$$

where the permutation of indices $\nu = 1, 2, 3$ by the operator \hat{P}_i is subject to the rule $\nu \bmod 3 = \nu$. It is obvious that the non-zero corrections have the components with $\mu = 0, \pm 2$. The denominator in the sum (4.33) renormalizes the interaction strength. The straightforward calculation of σ_μ and the use of Eq. (4.26) with the zero-order density matrix ρ gives $\chi/(1 + \chi\sigma_\mu) = 2\chi$. Combining (4.31) with (4.33), we have

$$\beta_n^{(\text{str})} = \frac{16\pi M^2 \omega_0^2}{15\hbar\omega_z \Sigma_z} \sum_{\mu=0,\pm 2} Q_{2\mu}^{(2)} Q_{2-\mu}^{(2)}. \quad (4.36)$$

We can now calculate the two contributions to the parameter β_n by summing over the quantum numbers n_x , n_y , and n_z . The anisotropic oscillator potential allows to find an exact solution of the problem. At first we find the corrections to the quadrupole moments

$$\begin{aligned} Q_{20}^{(2)} &= \sqrt{\frac{5}{64\pi}} \frac{\hbar \Sigma_z}{M\omega_0^2 \omega_z (k^2 - 1)} (2k^4 - 15k^2 + 1), \\ Q_{2\pm 2}^{(2)} &= \sqrt{\frac{5}{128\pi}} \frac{\hbar \Sigma_z (1 - 5k^2)}{M\omega_0^2 \omega_z k^2 (k^2 - 1)}. \end{aligned} \quad (4.37)$$

Then by using (4.36) we obtain the contribution of the centrifugal stretching effect

$$\beta_n^{(\text{str})} = \frac{\hbar \Sigma_z}{3\omega_z^3 k^4 (k^2 - 1)^2} (k^8 - 15k^6 + 76k^4 - 15k^2 + 1). \quad (4.38)$$

Finally, after some fairly tedious calculations of the sum (4.21) we get

$$\beta_n^{(\text{sp})} = \frac{\hbar \Sigma_z}{2\omega_z^3 k^4 (k^2 - 1)^2} (k^8 - 10k^6 - 14k^4 - 10k^2 + 1). \quad (4.39)$$

Adding the last two quantities gives us the parameter β in the normal phase:

$$\beta_n = \frac{5\mathfrak{S}_{\text{rig}}}{6\omega_0^2} \frac{k^4 - 10k^2 + 1}{k^{2/3}(k^2 + 1)}, \quad (4.40)$$

if we use the following formula for the rigid body moment of inertia:

$$\mathfrak{S}_{\text{rig}} = \frac{\hbar \Sigma_z}{\omega_z k^2} (k^2 + 1). \quad (4.41)$$

Parameter β_n is substantially reduced compared to β_s , $\beta_n \sim \hbar^4 A^{7/3} / \varepsilon_F^3 \sim \beta_s(\text{SD}) A^{-2/3}$. This can be explained by canceling main terms in the sums (4.21) and (4.35). That is exactly why the corresponding values $\beta_n^{(\text{sp})}$ and $Q_{2\mu}^{(2)}$ are proportional to Σ_z . Such result is predictable because the Hamiltonian h_ω for the anisotropic harmonic oscillator can be diagonalized exactly [30]. Its eigenstates are characterized by the number of rotating bosons. To find $\beta_n^{(\text{sp})}$ and $Q_{2\mu}^{(2)}$ we have to calculate first the expectation values of the operators ℓ_x and $Q_{2\mu}$ in this rotating basis. Then these quantities must be expanded in powers of ω . Because these operators are represented by quadratic forms in the rotating bosons, their mean values and therefore all the terms of the series are proportional to the linear combination of Σ_x , Σ_y , and Σ_z .

Another peculiarity of the solution (4.40) is that $\beta_n < 0$ for the prolate nuclei with $1 < b/a < 3.15$, whereas β_s is always positive. The formal cause of this effect is a negative value of $\beta_n^{(\text{sp})}$ and the inequality $|\beta_n^{(\text{sp})}| > \beta_n^{(\text{str})} > 0$ which is fulfilled for the above indicated deformations. In the superfluid phase, the term responsible for the rotation-quasiparticle interaction may be also negative, but it never exceeds the contribution of the pair modification effect (see Fig. 2a).

5 Analysis of experimental data

We have shown in the preceding section that the second inertial parameter \mathcal{B} is negative in the superfluid phase and positive in the normal one. The two limiting cases allow us to reconstruct the $\mathcal{B}(I)$ dependence for the parametrization (1.1) of a rotational sequence with $(\pi\alpha) = (+0)$. Comparing the formulas (4.10) or (4.15) with (4.40) we conclude that the ratio \mathcal{B}/\mathcal{A} has to change sign with increasing the spin I in a band and approach its limiting value $\mathcal{B}_n/\mathcal{A}_n \sim A^{-8/3}$ for high I .

The limiting ratio for a real nucleus can be obtained from Eqs. (4.40) and (4.41) if we suppose that the r.m.s radius and the deformation are exactly the same for neutron (ν) and proton (π) systems. The first condition implies that oscillator frequencies of neutron and proton potentials satisfy the relation $\omega_{0\tau} = \omega_0(2A_\tau/A)$ ($\tau = \pi, \nu$ and A_τ is the number of nucleons of a given type). The second one results in the identical ratio of the frequencies along the principal axes for the both potentials:

$$\omega_{x\tau} : \omega_{y\tau} : \omega_{z\tau} = m : m : l. \quad (5.1)$$

For integers m and l , the states with the same number of quanta $\mathcal{N}_{ml} = m(n_x + n_y) + ln_z$ form a deformed shell. Assuming that, for a given number of nucleons A_τ , all the \mathcal{N}_{ml} -shells are filled, one can express the sum $\Sigma_{z\tau}$ in the form

$$\omega_{z\tau}\Sigma_{z\tau} = \omega_{0\tau}(\Sigma_{x\tau}\Sigma_{y\tau}\Sigma_{z\tau})^{1/3} = \omega_{0\tau}\left(\frac{A_\tau^4}{32}\right)^{1/3}. \quad (5.2)$$

The above formulas allow us to derive the ratio \mathcal{B}/\mathcal{A} for a nucleus consisting of Z protons and N neutrons in the normal state:

$$\frac{\mathcal{B}_n}{\mathcal{A}_n} = -3.205 \frac{(k^4 - 10k^2 + 1)k^{2/3}}{(k^2 + 1)^3 A^{8/3}} \left[\left(\frac{Z}{A}\right)^{1/3} + \left(\frac{N}{A}\right)^{1/3} \right]. \quad (5.3)$$

This result holds for a nucleus with an arbitrary deformation $k = b/a = m/l$.

We concentrate first on SD bands. Most of them are not connected to lower-lying states of known excitation energy, spin, and parity. Thus, their exit spins I_0 are unknown. Tentative spin assignment is used to take advantage of the formulas (1.2) to find the experimental ratio \mathcal{B}/\mathcal{A} . To analyze this quantity, we will take into account two basic ingredients: shell gaps, which stabilize the shape, and intruder orbitals involved in the alignment. The nucleon-configuration assignment of a band is generally based only on the behavior of the dynamic moment of inertia and the quadrupole moment in a given band. The last quantity,

$$Q_0 = 6.05 \cdot 10^{-3} A^{2/3} \frac{k^2 - 1}{k^{2/3}} eb, \quad (5.4)$$

remains remarkably constant as a function of spin within a band. This proves that the deformation k remains practically unchanged as I increases. We use the experimentally observed value of Q_0 to find the axis ratio b/a , which is required for calculation of the quantity (5.3).

The ratio \mathcal{B}/\mathcal{A} extracted from the measured energy of γ -transitions in the four SD bands of the $A = 150$ mass region are shown in Fig. 4. The parity and the signature of these bands is assumed to be $(+, 0)$. We also use the adopted spins for their lowest levels. The

band $^{152}\text{Dy}(1)$ belongs to the doubly-magic nucleus with the proton $Z = 66$ and the neutron $N = 86$ gaps in single-particle spectra at the same deformation [31]. The gaps decrease a level density and considerably reduce the neutron and proton pairing correlations. There is no direct experimental indication of pairing correlations in this band. The theoretical calculations [32] show that their inclusion leads to a better description of the kinematic and dynamic moments of inertia, pairing correlations being more important at the low spin range. The plot shows that there are two distinct regions in the variation of the ratio \mathcal{B}/\mathcal{A} versus I . The lower part of the band exhibits a sharp increase of this quantity. It then changes sign at the spin $I_c = 36$ and approaches the plateau value of (5.3) at the top of the band. Such behavior of the ratio apparently shows that the static pairing correlations of neutrons and protons are quenched simultaneously. This fact also proves that most part of the band belongs to the normal phase.

The band $^{144}\text{Gd}(1)$ is one of the few examples of SD bands which exhibits backbending. The $\pi 6^2$ pair alignment opens up the proton shell gap $Z = 64$ at the same deformation as the neutron shell gap $N = 80$. Thus, above the backbending this band becomes similar to the doubly-magic $^{152}\text{Dy}(1)$ except the gap $Z = 64$ is less pronounced than the one at $Z = 66$. Besides, the neutron gap $N = 80$ is somewhat smaller than $N = 86$. These factors enhance a level density and favor pairing correlations. As seen in Fig. 4 the behavior of \mathcal{B}/\mathcal{A} for this band in the low- I region is the same as that for $^{152}\text{Dy}(1)$ if we scale the axis of ordinates by the factor two. Accordingly, the critical value is somewhat larger, $I_c = 38$.

The features observed at low spins in the dynamic moment of inertia of the band $^{150}\text{Gd}(1)$ have been explained in terms of consecutive alignments of the $\nu 7^2$ and $\pi 6^2$ pairs [31]. For the configuration $\pi 6^2 \nu 7^2$ all levels below the $Z = 64$ and $N = 86$ shell gaps are occupied. The former is found at slightly smaller deformation than the latter. This factor diminishes the neutron gap and enhances neutron pairing. The Woods-Saxon [31] and the relativistic mean field [33] calculations make evident that static proton and neutron pairing exist at low spins, $I < 48$ ($\hbar\omega < 0.55$ MeV). In addition, the full self-consistent HFB calculations with the particle number projection [32] show that the effect of pairing on the moments of inertia in $^{150}\text{Gd}(1)$ is about twice as important as in $^{152}\text{Dy}(1)$. It is apparent from Fig. 4 that the static pairing correlations in the band $^{150}\text{Gd}(1)$ are stronger than in $^{144}\text{Gd}(1)$. One should expect even more stronger pairing correlations in the newly discovered prolate deformed band $^{154}\text{Er}(2)$ [34], since the proton Fermi level at $Z = 68$ lies in the region of a high level density above the $Z = 66$ gap. The experimental \mathcal{B}/\mathcal{A} dependence of Fig. 4 is consistent with this prediction. It is seen that this ratio does not exceed the value -5×10^{-5} and does not show the plateau.⁷ The bump seen at $I = 44$ ($\omega = 0.57$ MeV) can be attributed to the alignment of a pair of $i_{13/2}$ protons in agreement with the calculations of Ref. [31]. Thus, the plots of Fig. 4 show the correlation of the spin dependence of the ratio \mathcal{B}/\mathcal{A} with the level density near the Fermi surface: the higher the level density, the stronger pairing correlations, and the less marked the plateau.

The high deformed (HD) bands in the $A = 190$ mass region are related to the $Z = 80$ and $N = 112$ shell gaps. Most of these bands have similar values of $\mathfrak{S}^{(2)}$ that exhibits a smooth rise as a function of rotational frequency. This rise is attributed to the gradual alignment, in the presence of static pairing correlations, of $i_{13/2}$ protons and $j_{15/2}$ neutrons. The calculations with pairing are able to reproduce the general trend seen in experiment. The bands $^{194}\text{Hg}(1)$ and $^{194}\text{Pb}(1)$ are of central importance because their spins, parities, and

⁷It is worth mentioning that the nonaxial band $^{154}\text{Er}(1)$ demonstrates the \mathcal{B}/\mathcal{A} dependence with the critical spin $I_c = 31$ and a long plateau.

excitation energies are known [35, 36]. The plots of Fig. 5 for these bands demonstrate the gradual rise of the \mathcal{B}/\mathcal{A} ratio that confirms the presence of static pairing correlations.

Now we turn our attention to ND bands. There are several bands of isotopes Er, Yb, and Hf in which the static neutron pairing gaps are predicted to collapse. However, the proton system still has strong pairing correlations. Accordingly, the \mathcal{B}/\mathcal{A} vs. I plot for these bands exhibits a sharp rise, but does not approach the plateau. The yrast band of ^{84}Zr is an exception. Because protons and neutrons occupy in this nucleus similar orbitals near the Fermi surface, quasiparticle alignments and the elimination of pairing gaps occur at similar spin. Besides, the deformed shell gaps at $Z = N = 38$ and the low moment of inertia favor the transition to the normal phase. Combination of these factors makes the pairing properties of the ND band of ^{84}Zr similar to those of SD bands in the $A = 150$ mass region. Fig. 6 plots, as a function of spin I , the \mathcal{B}/\mathcal{A} ratios for the ND and SD bands of this nucleus determined from the data of Refs. [37] and [38], respectively. In the SD band $^{84}\text{Zr}(1)$, static pairing correlations are quenched completely due to the high rotational frequency. This inference is supported by the coincidence of experimental points with the plateau $\mathcal{B}_n/\mathcal{A}_n$. It also becomes apparent from this figure that at high spins the experimental ratio for the ND band reaches the same plateau. The difference in the limiting value of $\mathcal{B}_n/\mathcal{A}_n$ due to the difference in deformations ($\beta = 0.43$ and 0.55 for the ND and SD bands, respectively) is insignificant. The low spin part ($I < 18$) of the ND band is compatible with the transitional nature of the γ -soft nucleus: small β and a noticeable triaxiality. The alignment of the two $g_{9/2}$ quasiprotons and the subsequent alignment of the two $g_{9/2}$ quasineutrons are clearly seen in Fig. 6 as the humps A and B. Beyond the second alignment, a striking change in deformation occurs in the interval of spins $I = 18 - 22$. After the spin $I = 24$ the rotational behavior is compatible with the rigid rotation of a high deformed axially symmetric nucleus [37].

The characteristic behavior of the ratio \mathcal{B}/\mathcal{A} with the critical spin I_c and the pronounced plateau have also been found in the SD bands of the $A = 150$ nuclei having configurations different from $(+, 0)$. The bands $^{152}\text{Tb}(2)$ [the $\pi[301]1/2$ hole in the ^{152}Dy SD core] and $^{153}\text{Ho}(3)$ [the ^{152}Dy SD core coupled to the $\pi[523]7/2$ orbital] show the \mathcal{B}/\mathcal{A} dependence similar to that of the $^{152}\text{Dy}(1)$ band. The pair of identical bands $^{150}\text{Gd}(2)$ and $^{151}\text{Tb}(1)$ have the dependence similar to one of $^{150}\text{Gd}(1)$. All these bands have somewhat higher values of \mathcal{B}/\mathcal{A} than do their $(+, 0)$ counterparts. This proves that an odd nucleon or a particle-hole excitation reduces pairing correlations due to the blocking effect. Such phenomenon is characteristic of the static pairing regime [19]. The SD bands from different mass regions, $^{132}\text{Ce}(1)$, $^{133}\text{Ce}(1)$, and $^{60}\text{Zn}(1)$, exhibit the same behavior of \mathcal{B}/\mathcal{A} . A strong configuration dependent effect is observed in the bands where the odd neutron is placed in the $j_{15/2}$ intruder orbitals. For such bands, the \mathcal{B}/\mathcal{A} ratio is positive for all spins. The examples include the bands $^{149}\text{Gd}(1)$ (configuration $\pi 6^2\nu 7^1$), $^{151}\text{Dy}(1)$ ($\pi 6^4\nu 7^1$), and $^{153}\text{Dy}(1)$ ($\pi 6^4\nu 7^3$). A single-particle degree of freedom seems to destroy the typical behavior of the ratio \mathcal{B}/\mathcal{A} . More efforts are needed to explain this interesting feature.

These numerous examples prove the universality of the transition from the superfluid to the normal phase for SD and ND bands. This universality can be represented, according to Ref. [6], by the effective rotational Hamiltonian,

$$H_{\text{eff}} = a\mathbf{I}^2 + (I/I_c - 1)b\mathbf{I}^4 + c\mathbf{I}^6, \quad (5.5)$$

which describes the states of the $(+, 0)$ band in the transition region. The parameters a , b , c , and the critical spin I_c are the subjects of the microscopic theory, which has to take

into account static, dynamic, and uniform pairing. Incorporating the critical spin, which can be found from the experimental plot \mathcal{B}/\mathcal{A} vs. I , this concept of the superfluid-to-normal transition is free from ambiguities characteristic of the approach based on a change of the single-particle spectra [4, 5].

Using the results of our analysis we can now explain why some SD bands have extremely regular rotational spectra. Figure 4 shows that the most part of the bands $^{152}\text{Dy}(1)$ and $^{144}\text{Gd}(1)$ belongs to the plateau with the ratio of the inertial parameters $\mathcal{B}/\mathcal{A} \sim 10^{-6}$. So does the whole of the SD band $^{84}\text{Zr}(1)$, for which this ratio is 10^{-5} . The plateau is the manifestation of the normal phase with the anomalous small ratio (5.3), $\mathcal{B}_n/\mathcal{A}_n \sim A^{-8/3}$. The above values for the SD bands agree with this estimation. Therefore the extreme regularity is explained by quenching the static pairing correlations in the lower parts of these bands. On the contrary, the transition in the yrast band of ^{84}Zr occurs in its upper part. Accordingly, the top of the band has the same properties. It is important to note that the bands in which proton and neutron pairing gaps are present [$^{154}\text{Er}(2)$ and all the SD bands in the $A = 190$ mass region] and the bands with proton pairing alone [$^{168}\text{Yb}(\text{yr})$ and $^{186}\text{Hf}(\text{yr})$] are regular to a lesser extent.

We should mention one more feature which requires further investigation. The down-sloping of the \mathcal{B}/\mathcal{A} dependence is observed at the top of $^{152}\text{Dy}(1)$, $^{84}\text{Zr}(1)$, and other SD bands with extremely high spins. Because the quantity $\mathcal{B}_n/\mathcal{A}_n$ is a decreasing function of the deformation k , it would be natural to explain this feature by the increase of the nuclear elongation due to the enormous centrifugal stretching at the end of these bands.

6 Conclusion

Despite the vast amount of data collected and various theoretical interpretations suggested, a detailed understanding of many properties of SD bands has yet to be achieved. Pairing correlations are just one example of such properties. The presence of static pairing in SD bands is usually established by studying the behavior of the dynamic moment of inertia $\mathfrak{I}^{(2)}$ as a function of the rotational frequency ω . A band crossing associated with a quasiparticle alignment leads to a impressive decrease in $\mathfrak{I}^{(2)}$ with ω or a hump in this dependence. This gives an indication that static pairing correlations are present in that part of a band where such irregularities occur.

In this paper, the investigation of pairing correlations is based on the spin dependence of the second inertial parameter \mathcal{B} . This quantity, which is proportional to the difference $\mathfrak{I}^{(1)} - \mathfrak{I}^{(2)}$ in the high- I limit, turned out to be a more sensitive measure of the change in pairing correlations than $\mathfrak{I}^{(2)}$. The new method requires spin-signature assignments of the band states. However, it gives more definite information about the superfluid-to-normal transition in a band. The most important results obtained in this paper can be summarized as follows:

(i) The exact semiclassical expression for the second inertial parameter in the superfluid phase has been found by taking into account the effect of rotation on the Cooper pairs in the gauge-invariant form. The presence of nonuniform pairing reduces the nonadiabatic effect of rotation. Its influence increases strongly at superdeformation. The nonuniform pairing allows one to find correctly the interesting physical limits for the second inertial parameter.

(ii) The limit of zero static pairing is of special interest. It permits the function $\mathcal{B}(I)$ to be reconstructed by interpolating between the values of \mathcal{B} in the superfluid and normal phases.

The anisotropic oscillator model calculations show that there are two distinct regions in the variation of the ratio \mathcal{B}/\mathcal{A} with I . The lower part of a band is characterized by a gradual decrease of pairing. Accordingly, being negative the ratio \mathcal{B}/\mathcal{A} exhibits a sharp increase. It then changes the sign at the spin I_c and approaches the positive value characteristic of the normal phase. The critical point I_c , $\mathcal{B}(I_c) = 0$, is a signature of the superfluid-to-normal transition. The transition manifests itself in the modification of the rotational spectrum of a band.

(iii) The experimental spin dependence of \mathcal{B}/\mathcal{A} indicates an agreement with the theoretical prediction and demonstrates the universality of the transition to the normal state. This agreement is not a trivial fact because our calculations are based on the simplest model of a nuclear potential and do not take into account pairing fluctuations in the normal phase. Nevertheless, the agreement is not accidental because the universal dependence of \mathcal{B}/\mathcal{A} on I has been observed for a large number of SD bands and some ND ones.

(iv) The universal spin dependence of \mathcal{B}/\mathcal{A} explains the extreme regularity of some SD bands. The characteristic feature of this dependence is the pronounced plateau in the upper part of a SD band ($I > I_c$) corresponding to the normal phase. The calculated ratio in this part of a band is extremely small, $\mathcal{B}/\mathcal{A} \sim A^{-8/3}$. Thus, the closer the critical point I_c to the exit spin I_0 , the more regular its rotational spectrum. The spectacular examples are the bands $^{144}\text{Gd}(1)$ and $^{152}\text{Dy}(1)$ having $\mathcal{B}/\mathcal{A} \sim 10^{-6}$.

(v) Some new features have been observed in the upper parts of SD bands. The investigation of this region, which is free from pairing correlations, is extremely important for our understanding of the microscopic structure at the superdeformed minimum.

Acknowledgments

This work is supported in part by the Russian Foundation for Basic Research, project no. 00-15-96590.

Appendix A. Solution of the integral equations for the nonuniform pairing field

The effect of rotation on pairing correlations is described by the first, $\Delta^{(1)}(\mathbf{r})$, and the second, $\Delta^{(2)}(\mathbf{r})$, corrections to the pairing field, which enter into Eq. (3.4). We have seen in Section 2 that the integral equations that have to be solved are of the general form (2.30) and (2.31) for even and odd corrections, respectively. It is convenient to introduce into these equations the operator $\dot{\ell}_x$ that is a function of the space coordinates only.

We start by considering the equation for $\Delta^{(1)}$. Using the relation $\hbar\dot{\ell}_{12}^x = i(p_1 - p_2)\ell_{12}^x$ we get from (2.31) for $i = 0$

$$\sum_{1,2} \int_{C'} \frac{dp}{2\pi} \frac{1}{(\mathbf{p}_1^2 + \Delta^2)(\mathbf{p}_2^2 + \Delta^2)} [2i\Delta\hbar\omega\dot{\ell}_{12}^x + (p_1 - p_2)^2\Delta_{12}^{(1)}] \varphi_1(\mathbf{r})\varphi_2^*(\mathbf{r}') = 0. \quad (\text{A.1})$$

The equation is satisfied if we assume that

$$\Delta^{(1)}(\mathbf{r}) = -i\frac{\hbar\omega}{2\Delta} D_1 \dot{\ell}_x, \quad (\text{A.2})$$

where the amplitude D_1 is determined after substituting (A.2) into Eq. (A.1) and integrating over \mathbf{r} :

$$D_1 = 4\Delta^2 \sum_{1,2} |\dot{\ell}_{12}^x|^2 \int_{C'} \frac{dp}{2\pi} \frac{1}{(\mathbf{p}_1^2 + \Delta^2)(\mathbf{p}_2^2 + \Delta^2)} \Big/ \sum_{1,2} |\dot{\ell}_{12}^x|^2 \int_{C'} \frac{dp}{2\pi} \frac{(p_1 - p_2)^2}{(\mathbf{p}_1^2 + \Delta^2)(\mathbf{p}_2^2 + \Delta^2)}. \quad (\text{A.3})$$

The solution transforms to

$$D_1 = \frac{\sum_{1,2} |\dot{\ell}_{12}^x|^2 g(x_{12}) \delta(\varepsilon_1 - \varepsilon_F)}{\sum_{1,2} |\dot{\ell}_{12}^x|^2 x_{12}^2 g(x_{12}) \delta(\varepsilon_1 - \varepsilon_F)} \quad (\text{A.4})$$

in the semiclassical approximation. The summation over the state 1 is to be understood as the integration over its quantum numbers. For the anisotropic oscillator potential, the amplitude can be expressed in a simple analytical form

$$D_1 = \frac{g(\nu_1) + g(\nu_2)}{\nu_1^2 g(\nu_1) + \nu_2^2 g(\nu_2)}. \quad (\text{A.5})$$

The function $g(x)$ and the parameters ν_1 and ν_2 are determined by Eqs. (3.16) and (4.2), respectively.

The equation for $\Delta^{(2)}(\mathbf{r})$ after introducing $\dot{\ell}_x$ becomes

$$\begin{aligned} & \sum_{1,2,3} \left[2\hbar^2 \omega^2 \dot{\ell}_{12}^x \dot{\ell}_{23}^x \int_{C'} \frac{dp}{2\pi} \frac{\mathcal{Q}_3(p, p_1, p_2, p_3)}{(\mathbf{p}_1^2 + \Delta^2)(\mathbf{p}_2^2 + \Delta^2)(\mathbf{p}_3^2 + \Delta^2)} \right. \\ & \left. - \Delta_{12}^{(2)} \delta_{23} \int_{C'} \frac{dp}{2\pi} \frac{(p_1 - p_2)^2 + 4\Delta^2}{(\mathbf{p}_1^2 + \Delta^2)(\mathbf{p}_2^2 + \Delta^2)} \right] \varphi_1(\mathbf{r}) \varphi_2^*(\mathbf{r}') = 0, \end{aligned} \quad (\text{A.6})$$

where the polynomial function \mathcal{Q}_3 of the third order in (p, p_ν) depends also on the amplitude D_1 . We try to solve this equation by making substitution

$$\Delta^{(2)}(\mathbf{r}) = \frac{\hbar^2 \omega^2}{4\Delta^3} D_2 \dot{\ell}_x^2. \quad (\text{A.7})$$

Applying the same procedure as before, one can find the amplitude D_2 in the semiclassical approximation

$$D_2 = \frac{\sum_{1,2,3} \dot{\ell}_{12}^x \dot{\ell}_{23}^x (\dot{\ell}_x^2)_{31} \phi(x_{12}, x_{13}, x_{23}) \delta(\varepsilon_1 - \varepsilon_F)}{\sum_{1,2,3} |(\dot{\ell}_x^2)_{12}|^2 h(x_{12}) \delta(\varepsilon_1 - \varepsilon_F)}, \quad (\text{A.8})$$

where the function $h(x)$ is determined by (3.16) and that of ϕ has the form

$$\begin{aligned} \phi(x, y, z) &= \frac{1}{2x^2 y^2 z^2} [-xy(1 - D_1 x^2)(1 + xy - D_1 z^2)g(x) \\ &+ y^2(1 - D_1 x^2 - D_1 z^2)h(y) - yz(1 - D_1 z^2)(1 + yz - D_1 x^2)g(z)]. \end{aligned} \quad (\text{A.9})$$

Their symmetry properties are

$$\phi(z, y, x) = \phi(x, y, z), \quad \phi(-x, -y, -z) = \phi(x, y, z). \quad (\text{A.10})$$

In the oscillator potential, the sum over the states 2 and 3 in the numerator of (A.8) comprises 16 terms including four with two close transitions, four with two distant transitions, and eight

terms with one close and one distant transitions. Performing summation in the semiclassical approximation in the numerator and in the denominator of (A.8), we find

$$D_2 = [4\phi(\nu_1, \nu_1 - \nu_2, -\nu_2) + 4\phi(\nu_1, \nu_1 + \nu_2, \nu_2) + \phi(\nu_1, 2\nu_1, \nu_1) + \phi(\nu_2, 2\nu_2, \nu_2) + 4\phi(\nu_1, 0, -\nu_1) + 4\phi(\nu_2, 0, -\nu_2)]\mathcal{H}^{-1}(\nu_1, \nu_2), \quad (\text{A.11})$$

where

$$\mathcal{H}(\nu_1, \nu_2) = 8 + 4h(\nu_1 - \nu_2) + 4h(\nu_1 + \nu_2) + h(2\nu_1) + h(2\nu_2). \quad (\text{A.12})$$

For the monopole pairing interaction, the pairing field is uniform and the first correction $\Delta^{(1)}$ vanishes. The coordinate independent solution for the correction $\Delta^{(2)}$ can be found from Eq. (A.6) after averaging over \mathbf{r} . The resulting expression may be written in terms of the kinematic moment of inertia,

$$\Delta^{(2)} = \frac{\omega^2}{2\rho_F} \frac{\partial \mathfrak{S}^{(1)}}{\partial \Delta}, \quad \mathfrak{S}^{(1)} = \sum_{1,2} |\ell_{12}^x|^2 [1 - g(x_{12})] \delta(\varepsilon_1 - \varepsilon_F), \quad (\text{A.13})$$

in agreement with the result obtained by Marshalek [13]. From the theoretical viewpoint, this solution is not correct because it violates the current conservation.

Appendix B. Calculation of integrals

In this Appendix we give a brief outline of the technique used in calculation of the integrals which are necessary for obtaining the function F (3.12) and for the solution of the integral equations (A.1) and (A.6). All the relevant integrals can be done exactly with the method proposed by Feynman in the quantum electrodynamics [28]. The method is based on the identity

$$\frac{1}{a_1 a_2 \dots a_n} = (n-1)! \int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-2}} \frac{dt_{n-1}}{[a_1 t_{n-1} + a_2 (t_{n-2} - t_{n-1}) + \dots + a_n (1 - t_1)]^n}, \quad (\text{B.1})$$

which is proved by a direct calculation.

The simplest integral is the one involved in the sum (3.10). It is solved by using (B.1) as follows:

$$J_1 = \int \frac{d\mathbf{p}_1}{2\pi} \frac{1}{(\mathbf{p}_1^2 + \Delta^2)(\mathbf{p}_2^2 + \Delta^2)} = \int_0^1 dt \int \frac{d\mathbf{p}_1}{2\pi} \frac{1}{[\mathbf{p}_1^2 + p_{12}^2 Q(t)]^2} = \frac{1}{2p_{12}^2} \int_0^1 \frac{dt}{Q(t)} = \frac{1}{2\Delta^2} g\left(\frac{p_{12}}{2\Delta}\right), \quad (\text{B.2})$$

where

$$Q(t) = -t^2 + t + \delta^2, \quad \delta = \Delta/p_{12}, \quad p_{12} = p_1 - p_2.$$

Four integrals appear in the first sum of Eq. (A.6). All those are of the same type. As an example, we consider the term proportional to the square of the amplitude D_1 . The relevant integral is

$$J_2 = \Delta \int \frac{d\mathbf{p}_1}{2\pi} \frac{\mathcal{Q}_2(p; p_1, p_2, p_3)}{(\mathbf{p}_1^2 + \Delta^2)(\mathbf{p}_2^2 + \Delta^2)(\mathbf{p}_3^2 + \Delta^2)}$$

$$= \Delta \int_0^1 dt_1 \int_0^{t_1} dt_2 \int \frac{d\mathbf{p}_1}{2\pi} \frac{\mathcal{Q}_2(p; p_1 + (t_1 - t_2)p_{12} + (1 - t_1)p_{13})}{[\mathbf{p}_1^2 + p_{13}^2 Q(t_1, t_2)]^3}, \quad (\text{B.3})$$

where $\mathcal{Q}_2 = p^2 + p_1 p_2 - p_1 p_3 + p_2 p_3 + \Delta^2$,

$$Q(t_1, t_2) = -[(1 - c^2)t_1 + c^2 t_2 - 1]^2 - (1 - c^2)t_1 - c^2 t_2 + 1 + \delta^2,$$

and $c = p_{12}/p_{13}$. By making the substitution $u_1 = (1 - c)t_1 + ct_2$, $u_2 = t_2$, we get after integration over $d\mathbf{p}_1$

$$J_2 = \frac{\Delta}{2p_{13}p_{23}} \left\{ \int_0^{1-c} du_1 \int_0^{u_1} du_2 + \int_{1-c}^1 du_1 \int_{(u_1-1+c)/c}^{u_1} du_2 \right\} \frac{(1-c)u_1 - cu_2 - 1 + c - 2\delta^2}{[cu_2 + Q_1(u_1)]^2}, \quad (\text{B.4})$$

where $Q_1(u_1) = -u_1^2 + (1 - c)u_1 + \delta^2$. Straightforward calculation of these integrals gives

$$J_2 = \frac{1}{2\Delta x_{13}} [x_{12}g(x_{12}) + x_{23}g(x_{23})]. \quad (\text{B.5})$$

Finally, let us consider the integral (3.9). It is solved in the same manner as the preceding ones. After using the identity (B.1) and integration over $d\mathbf{p}_1$, the substitution $t_1 = [u_1 - (d - c)u_2 - cu_3]/(1 - d)$, $t_2 = u_2$, $t_3 = u_3$, where $c = p_{12}/p_{14}$, $d = p_{13}/p_{14}$, leads to four triple integrals which may be solved without a problem.

References

- [1] I. Gurevich, Zh. Éksp. Teor. Fiz. **9**, 1283 (1939).
- [2] B. R. Mottelson and J. G. Valatin, Phys. Rev. Lett. **5**, 511 (1960).
- [3] Y. R. Shimizu *et al.*, Rev. Mod. Phys. **61**, 131 (1989).
- [4] J. D. Garrett, G. B. Hagemann, and B. Herskind, Ann. Rev. Nucl. Part. Sci. **36**, 419 (1986).
- [5] J. R. B. Oliveira, *et al.*, Phys. Rev. C **47**, R926 (1993).
- [6] I. M. Pavlichenkov, Phys. Reports **226**, 175 (1993).
- [7] I. M. Pavlichenkov, Zh. Éksp. Teor. Fiz. **96**, 404 (1989) [Sov. Phys. JETP **96**, 227 (1989)].
- [8] B. Singh, R. B. Firestone, and S. Y. Frank Chy, Nucl. Data Sheets **78**, 1 (1996).
- [9] A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1975), Vol. II.
- [10] I. N. Mihailov, K. Neergård, V. V. Pashkevich, and S. Frauendorf, Fiz. Élem. Chastits At. Yadra **8**, 1338 (1977) [Sov. J. Part. Nucl. **8**, 550 (1977)].
- [11] Yu. T. Grin' and I. M. Pavlichenkov, Zh. Éksp. Teor. Fiz. **43**, 465 (1962) [Sov. Phys. JETP **16**, 333 (1963)].

- [12] I. M. Pavlichenkov, Nucl. Phys. **55**, 225 (1964).
- [13] E. R. Marshalek, Phys. Rev. **158**, 993 (1967).
- [14] I. Dabrowski, Can. J. Phys. **62**, 1639 (1984).
- [15] W. Satuła and R. Wyss, Phys. Rev. C **50**, 2888 (1994).
- [16] J. Dobaczewski and J. Dudek, Phys. Rev. C **52**, 1827 (1995).
- [17] P.-H. Heenen *et al.*, Nucl. Phys. **A598**, 169 (1996).
- [18] Yang-Sun *et al.*, Phys. Rev. Lett. **78**, 2321 (1997).
- [19] Y. R. Shimizu, Nucl. Phys. **A520**, 477c (1990).
- [20] I. Hamamoto and W. Nazarewicz, Phys. Rev. C **49**, 2489 (1994).
- [21] A. B. Migdal, Nucl. Phys. **13**, 655 (1959).
- [22] I. M. Pavlichenkov, JETP Lett. **71**, 4 (2000).
- [23] I. M. Pavlichenkov, Yad. Fiz. **64**, 1168 (2001) [Phys. At. Nucl. **64**,1095 (2001)].
- [24] L. P. Gor'kov, Zh. Éksp. Teor. Fiz. **36**, 1918 (1959) [Sov. Phys. JETP **9**, 1364 (1959)].
- [25] S. T. Belyaev, Yad. Fiz. **4**, 936 (1966) [Sov. J. Nucl. Phys **4**, 671 (1966)].
- [26] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinski, *Quantum Field Theoretical Methods in Statistical Physics* (Pergamon, Oxford, 1965).
- [27] K. Mühlhans, M. Diebel, K. Neergård, and U. Mosel, Phys. Lett. B **120**, 44 (1983).
- [28] R. P. Feynman, Phys. Rev. **76**, 769 (1949).
- [29] H. Sakamoto and T. Kishimoto, Nucl. Phys. **A501**, 205 (1989).
- [30] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, New York, 1980).
- [31] W. Nazarewicz, R. Wyss, and A. Johnson, Nucl. Phys. **A503**, 285 (1989).
- [32] N. El Aouad *et al.*, Nucl. Phys. **A676**, 155 (2000).
- [33] A. V. Afanasjev, G. A. Lalazissis, and P. Ring, Nucl. Phys. **A634**, 393 (1998).
- [34] K. Lagergren, *et al.*, Phys. Rev. Lett. **87**, 022502 (2001).
- [35] G. Hackman, *et al.*, Phys. Rev. Lett. **79**, 4100 (1997).
- [36] K. Hauschild, *et al.*, Phys. Rev. C **55**, 2819 (1997).
- [37] H. G. Price, *et al.*, Phys. Rev. Lett. **51**, 1842 (1983).
- [38] H.-Q. Jin, *et al.*, Phys. Rev. Lett. **75**, 1471 (1995).

FIGURE CAPTIONS

Fig. 1. Relative deviation of energies E in the superdeformed band $^{194}\text{Pb}(1)$ as compared with the ground state band of ^{238}U and the ground vibrational state band of the H_2 molecule. The deviation is calculated from the formula $(E - E_{\text{rig}})/E_{\text{rig}}$, where $E_{\text{rig}} = \mathcal{A}I(I+1)$, and the parameters \mathcal{A} are found from the energies $E_\gamma(4)$ of the $6 \rightarrow 4$ transitions. The experimental data are taken from Refs. [8] and [14].

Fig. 2. Comparison of the functions relevant to the second inertial parameter for ND bands. Part (a) represents the function Φ_n (4.5) (solid line), Φ_{nc} (4.11) (dashed line), and their parts, the functions \mathcal{F} (4.6) (dotted line) and \mathcal{F}_c (4.12) (dash-dotted line) which describe the effect of rotation-quasiparticle interaction. Part (b) is for the limit of close transitions. The function Φ_{nc} (solid line) and its constituents \mathcal{F}_c (dashed line) and $2D_2^2\mathcal{H}_c$ (dotted line) are shown for $\Delta^{(1)} = 0$. The axis of abscissas shows the dimensionless quantity ν_1 corresponding to the close transitions, while that for the distant ones is fixed by the representative value $\nu_2 = 10$ for all the plots.

Fig. 3. Plot of the function Φ , to which the second inertial parameter for SD bands (4.15) is proportional, against the dimensionless quantity ξ for the axis ratio $b/a = 2$. The solid and dashed lines correspond to the exact value and the limit of uniform pairing, respectively. The abscissa scale must be multiplied by a factor of approximately 7.7 for nuclei in the $A \sim 150$ mass region in order to obtain the gap energy in MeV.

Fig. 4. Ratio \mathcal{B}/\mathcal{A} versus spin for some SD bands of the $A = 150$ region. Expressions (1.2) are used to extract this ratio from experimental data taken from Refs. [8] and [34]. The error bars (if they are greater than symbol sizes) include only the uncertainties in γ -ray energies. The uncertainties in the spin assignment are immaterial for all bands [with the exception of $^{152}\text{Dy}(1)$], since the spin variation in $2\hbar$ would merely shift the curves along the abscissa. The experimental points for the band $^{144}\text{Gd}(1)$ are shown above the $\pi i_{13/2}$ backbending. The solid straight line is the ratio $\mathcal{B}_n/\mathcal{A}_n$ (5.3) for the normal phase with the deformation b/a found from the quadrupole moment (5.4).

Fig. 5. The same as in Fig. 4 for the two SD bands of the $A = 190$ region. Experimental data are taken from Refs. [8, 35, 36].

Fig. 6. Ratio \mathcal{B}/\mathcal{A} versus spin for the yrast ND band (solid circles) and the SD band (open circles) of ^{84}Zr . Experimental data are taken from Refs. [37] and [38]. The solid straight line is the ratio $\mathcal{B}_n/\mathcal{A}_n$ (5.3) relevant to the deformation of the SD band.